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KEY

TO THE

ELEMENTS OF ALGEBRA.

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LONDON: JOHN WEALE.

A

KEY AND COMPANION

TO THE

RUDIMENTARY ALGEBRA:

FORMING AN EXTENSIVE REPOSITORY OF

SOLVED EXAMPLES AND PROBLEMS,

IN ILLUSTRATION OF THE VARIOUS EXPEDIENTS NECESSARY IN ALGEBRAICAL OPERATIONS

ESPECIALLY ADAPTED FOR SELF-INSTRUCTION.

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PREFACE.

To a work of this kind very few observations need be prefixed by way of Preface; it may be proper to mention, however, that it was undertaken to accomplish a two-fold purpose.

The publisher of the Rudimentary Treatises conceived that a book of Questions and Examples, exhibiting in detail the various artifices and expedients which algebraists employ to facilitate their operations, more especially in the solution of complicated equations of the first and second degree, might form a useful companion to existing elementary works on Algebra.

The author concurred in this opinion; and as Mr. Haddon's Algebra is distinguished for the great number of unsolved exercises which it contains,—many of them of an order of difficulty above the powers of an unaided student,—he considered that two desirable objects would be attained by the publication of a KEY and COMPANION to that work.

In those places of education where the Algebra is used, the KEY cannot fail to be very acceptable to the teacher; and, as a copious collection of carefully solved examples, it is calculated to furnish, at a very moderate price, much necessary aid and instruction to the young algebraist, whatever text-book he may apply himself to.

Mr. Haddon's problems and exercises seem to have been selected from a variety of sources; and it is pro-

bable that the solutions to most of them already exist. But the author thinks it right to mention that those here given are his own, with perhaps two, or at most three exceptions. He conceives that some of the expedients he has adopted, in unravelling the intricacies of certain equations of the second degree, have claim to novelty, and, indeed, to a character of generality that renders them applicable beyond the limits of the particular cases to which they are specially applied.

The work, though small in size, is the result of a good deal of thought and labour; both of which might, no doubt, have been greatly diminished by searching for solutions through the various existing treatises on Algebra. But to this kind of mechanical drudgery the author felt a decided repugnance; and he was moreover anxious, if possible,—by resisting all temptation to copy from others,—to give to his solutions some traits of originality, and so, perchance, to soften the asperities of what the learner might regard as some of the harsher features of elementary algebra.

ERRATA.

Page	4	line	13	*for*	$=a$	*read*	$=2a$
,,	5	,,	8	,,	a^3b	,,	a^2b
,,	19	,,	23	,,	$2x$	,,	$9x$
,,	21	,,	17	,,	$(a-2b^2)$	,,	$(a-2b)^2$.
,,	27	,,	22	,,	$(x-y)^4$	,,	$(x+y)^4$
,,	64	,,	17	,,	10	,,	56
,,	64	,,	24	,,	10	,,	60
,,	67	,,	17	,,	$5x$	,,	5^x
,,	70	,,	6	,,	$x-{}^{-\frac{3}{2}}$	,,	$x^{-\frac{3}{2}}$
,,	75	,,	7	,,	10	,,	56

,, 80 Ex. 81, supply the exponent $\frac{1}{2}$

,, 84 Ex. 90, supply the exponent $\frac{3}{2}$

,, 92 line 11 *for* + *read* −

,, 95 Ex. 15, ,, $y-b$,, $y=b$

,, 103 Ex. 39, supply $\sqrt{}$ before a_2-4b^n

,, 113 line 16 *for* b *read* $b^{\frac{1}{3}}$

,, 120 ,, 3 ,, 115 ,, 105

,, 138 ,, 12 ,, $n=1$,, $n-1$

KEY AND COMPANION

TO THE

RUDIMENTARY ALGEBRA.

THE first rule in Algebra, called ADDITION, differs from the corresponding rule in Arithmetic chiefly in this: that whereas, in the latter science *all* the quantities concerned are really added together, in the former science some of them are frequently to be subtracted. Those quantities, among the entire set that are to be subtracted, are always preceded by the subtractive sign *minus;* that is to say, by the mark —.

Addition of Algebra, therefore, means the finding the balance of a set of quantities, when some of them are additive and others subtractive.

Different letters in Algebra stand, of course, for different things, and hence the distinction between like and unlike quantities (Algebra, p. 3). But the learner is not to understand that the unlike quantities, connected together by the signs *plus* and *minus,* in any of the following examples, stand for things altogether different in *kind;* to admit of being added or subtracted, whether in arithmetic or algebra, the things represented by the figures or letters must necessarily be of the same kind; the different letters merely imply difference of *value,* not any difference in the nature of the things themselves; in general, the things are simply abstract numbers.

For instance, the expression $3a + 2b$, in example 1 in the book, means 3 times the quantity represented by a (whatever

it be) increased by twice the quantity of the same kind, but of different value, represented by b. We cannot actually add the $3a$ to the $2b$, till the numerical values of a and b are interpreted. Previously to this, the addition can only be indicated by connecting the quantities by the *plus* sign, as above; but when the letters are the same, we can execute the addition implied in the sign: for we know that $3a + 2a$ must be $5a$, whatever a may stand for.

It is thus that, in the following examples, the *like* quantities can always be actually added or incorporated, or the balance of the additive and subtractive like quantities expressed by a single term; while the addition or subtraction of quantities which have no like, can be *indicated* only by the *plus* or *minus* sign.

Addition (Page 5).

Arranging the quantities to be added, like under like, the examples in the book will stand as follows:—

(Ex. 4.)
$$\begin{array}{r} 2a+2b \\ a+3b \\ 5a+\ b \\ 8a+\ b \\ \hline 16a+7b \end{array}$$

(5.)
$$\begin{array}{r} 2x-4y \\ -3x+\ y \\ 6x-5y \\ -x+2y \\ \hline 4x-6y \end{array}$$

(6.)
$$\begin{array}{r} 2x^2+\ xy-2y^2 \\ -4x^2+3xy-\ y^2 \\ -\ x^2-6xy+5y^2 \\ 4x^2-\ xy+3y^2 \\ \hline x^2-3xy+5y^2 \end{array}$$

(7.)
$$\begin{array}{r} ax^3+\ bx^2-\ cx \\ 2ax^3-5bx^2+4cx \\ -ax^3+2bx^2-8cx \\ \hline 2ax^3-2bx^2-5cx \end{array}$$

(8.)
$$\begin{array}{r} a+b \\ a-b \\ \hline 2a \end{array} \qquad \begin{array}{r} x^2+xy \\ xy+y^2 \\ \hline x^2+2xy+y^2 \end{array} \qquad \begin{array}{r} y^2-xz \\ xz-z^2 \\ \hline y^2-z^2 \end{array}$$

Note.—In adding a set of quantities together, it is plain that the order in which they are written down for this purpose is of no moment, except in so far as convenience may be served.

In some of the preceding examples, the order in which the individual *terms* of an *expression* succeed one another has been changed, for the convenience of arranging like quantities in the same vertical column.

It may be mentioned here, that any algebraical quantity is called an *expression*, and the individual parts into which it is separated by the signs $+$, $-$, are called the *terms* of that expression: thus, $2a$ is an expression of one term, $2a+3b$ is an expression of two terms, $2x+xy-2y^2$, an expression of three terms, &c.

SUBTRACTION (Page 6).

In subtraction, we have only to imagine the signs of all the quantities to be subtracted to be *changed*, and then to proceed as in addition. The learner must be careful to regard all the signs in the row of terms to be subtracted as thus changed, and if any term in this row have no like in the row above, to bring it down in the remainder with changed sign.

$$\begin{array}{rr} (4.)\ & 5a-6b-3c \\ & 2a+3b-7c \\ \hline \text{Remainder} & 3a-9b+4c \end{array} \qquad \begin{array}{r} 5a-6b-3c \\ -2a-3b+7c \\ \hline 3a-9b+4c \end{array}$$

Or by actually changing the signs in the lower row of terms, and adding, the work will stand as on the right.

In the first of these forms, the signs of the lower row of terms are *conceived* to be changed; in the second, the change is actually made. This change of signs converts the subtraction into addition.

$$\begin{array}{rr} (5.)\ & x^2-xy \\ & xy-y^2 \\ \hline \text{Rem.} & x^2-2xy+y^2 \end{array} \qquad \begin{array}{rr} & a+b \\ & a-b \\ \hline \text{Rem.} & 2b \end{array}$$

$$\begin{array}{rr} (6.)\ & 5n^2+n-3 \\ & 4n^2-3n-2 \\ \hline \text{Rem.} & n^2+4n-1 \end{array} \qquad \begin{array}{rr} (7.)\ & ay^2-7a^2y-a-c \\ & 3ay^2-2a^2y+a-b \\ \hline \text{Rem.} & -2ay^2-5a^2y-2a+b-c \end{array}$$

$$
\begin{array}{rl}
(8.) & a^3-a^2b+2ab^2-c \\
 & \quad 2a^2b-ab^2 \\
\hline
\text{Rem.} & a^3-3a^2b+3ab^2-c. \\
\hline
\end{array}
$$

By adding the remainder to the row of terms immediately above it, the result will be the top row, whenever the operation is correct.

NOTE.—From example 8 in Addition, and example 5 in Subtraction, we see that if the difference of two quantities be added to the sum, the result will be twice the greater of those quantities; and that if the difference be subtracted instead of added, the result will be twice the less: for those examples show that

$$(a+b)+(a-b)=a, \text{ and that } (a+b)-(a-b)=2b,$$

whatever numbers, or quantities of the same kind, a and b may stand for. Suppose, for instance, that a stood for 13, and b for 5: then the sum would be 18, and the difference 8; and $18+8=26$—that is, twice 13, the greater; also $13-8=10$—that is, twice 5, the less. The learner will find it useful to keep this general principle in remembrance.

MULTIPLICATION (Page 9).

(3). $6a \times 4=24a$; $3a \times 5b=15ab$; $x^4 \times x^2=x^{4+2}=x^6$.

(4.) $(2x-4y+z) \times 3x=6x^2-12xy+3xz$;
$(a^2+2ab-b^2) \times a^3b^2=a^5b^2+2a^4b^3-a^3b^4$.

NOTE.—When, as in this last example, one of the *factors* is a compound expression—that is, an expression made up of two or more simple terms—it is necessary to inclose the terms of the compound factor in a *vinculum*, or to bind them together in one whole, as above, in order to indicate that that *whole* is to be multiplied. If we had written the first of the examples marked (4) thus: $2x-4y+z \times 3x$, without any tie connecting the terms $2x-4y+z$ into one whole, we should have indicated that the z *only* is to be multiplied by $3x$, and *not* the other terms $2x-4y$; but by means of the vinculum, ( ), or [ ], or { }, &c., we imply that *all* the terms are to be equally multiplied by $3x$.

(5.)
$$\begin{array}{l}
x^3-x^2y+xy^2-y^3\\
x+y\\
\hline
x^4-x^3y+x^2y^2-xy^3\\
\quad\quad x^3y-x^2y^2+xy^3-y^4\\
\hline
x^4 \qquad\qquad\qquad\qquad -y^4\\
\hline
\end{array}$$

$$\begin{array}{l}
a^2+b^2+c^2+ab-ac+bc\\
a-b+c\\
\hline
a^3+ab^2+ac^2+a^2b-a^2c+abc\\
\quad -a^2b-b^3-bc^2-ab^2+abc-b^2c\\
\quad\quad\quad a^2c+b^2c+c^3+abc-ac^2+bc^2\\
\hline
a^3 \qquad -b^3 \qquad +c^3+3abc\\
\hline
\end{array}$$

Consequently, $(x^3-x^2y+xy^2-y^3)\ (x+y)=x^4-y^4$, and

$(a^2+b^2+c^2+ab-ac+bc)\ (a-b+c)\ =a^3-b^3+c^3+3abc.$

(6.)
$$\begin{array}{l}
5x^3+4x^2+3x+2\\
5x^3-4x^2\\
\hline
25x^6+20x^5+15x^4+10x^3\\
\quad\quad -20x^5-16x^4-12x^3-8x^2\\
\hline
25x^6 \qquad - \quad x^4- \ 2x^3-8x^2.\\
\hline
\end{array}$$

(7.)
$$\begin{array}{l}
a+b\\
a+b\\
\hline
a^2+ab\\
\quad\ ab+b^2\\
\hline
a^2+2ab+b^2\\
\hline
\end{array}
\qquad\qquad
\begin{array}{l}
a-b\\
a-b\\
\hline
a^2-ab\\
\quad -ab+b^2\\
\hline
a^2-2ab+b^2\\
\hline
\end{array}
\qquad\qquad
\begin{array}{l}
a+b\\
a-b\\
\hline
a^2+ab\\
\quad -ab-b^2\\
\hline
a^2 \qquad -b^2\\
\hline
\end{array}$$

Note.—As recommended in the book, the products in example 7 should be kept in the memory. The learner should be able to write down the results of such expressions as $(x+y)^2$, $(x-y)^2$, $(x+y)\ (x-y)$ at once, without any work: thus, from the preceding model,

$(x+y)^2=x^2+2xy+y^2$, $(x-y)^2=x^2-2xy+y^2$, $(x+y)(x-y)=x^2-y^2$,

where x and y merely take the places of a and b above.

Division (Page 11).

$$
\begin{array}{l}
(9.)\ 4x+2a)\ 8x^2+16ax+6a^2\ (2x+3a \\
\qquad\qquad\quad 8x^2+4ax \\
\hline
\qquad\qquad\qquad\quad 12ax+6a^2 \\
\qquad\qquad\qquad\quad 12ax+6a^2
\end{array}
$$

$$
\begin{array}{l}
x^2+2xy+y^2)\ x^4+x^3y+xy^3+y^4\ (x^2-xy+y^2 \\
\qquad\qquad\qquad x^4+2x^3y+x^2y^2 \\
\hline
\qquad\qquad\qquad -x^3y-x^2y^2+xy^3 \\
\qquad\qquad\qquad -x^3y-2x^2y^2-xy^3 \\
\hline
\qquad\qquad\qquad\qquad x^2y^2+2xy^3+y^4 \\
\qquad\qquad\qquad\qquad x^2y^2+2xy^3+y^4
\end{array}
$$

Note.—Whenever dividend and divisor have a factor in common, we may always suppress the common factor before commencing the operation; because the quotient must be the same, whether we divide one quantity by another, or half, a third, a fourth, &c., of the former, by half, a third, a fourth, &c., of the latter. In the first of the above examples, 2 is a factor common to all the terms of dividend and divisor, so that this factor may be suppressed, and thus the half only of each taken as follows:—

$$
\begin{array}{l}
(9.)\ 2x+a)\ 4x^2+8ax+3a^2\ (2x+3a \\
\qquad\qquad\quad 4x^2+2ax \\
\hline
\qquad\qquad\qquad\quad 6ax+3a^2 \\
\qquad\qquad\qquad\quad 6ax+3a^2
\end{array}
$$

$$
\begin{array}{l}
(10.)\ 4x^2y+3xy-1)\ 8x^4y+2x^3y-2x^2-3x^2y+x\ (2x^2-x \\
\qquad\qquad\qquad\qquad 8x^4y+6x^3y-2x^2 \\
\hline
\qquad\qquad\qquad\qquad\quad -4x^3y-3x^2y+x \\
\qquad\qquad\qquad\qquad\quad -4x^3y-3x^2y+x
\end{array}
$$

$$
\begin{array}{l}
(11.)\ x-y)\ x^4-y^4\ (x^3+x^2y+xy^2+y^3 \\
\qquad\qquad x^4-x^3y \\
\hline
\qquad\qquad\quad x^3y-y^4 \\
\qquad\qquad\quad x^3y-x^2y^2 \\
\hline
\qquad\qquad\qquad x^2y^2-y^4 \\
\qquad\qquad\qquad x^2y^2-xy^3 \\
\hline
\qquad\qquad\qquad\quad xy^3-y^4 \\
\qquad\qquad\qquad\quad xy^3-y^4. \\
\hline
\end{array}
$$

NOTE.—This last operation may be conducted differently by aid of the theorem 3 at page 9 (Algebra), and the importance of which has been mentioned above. For by that theorem, $x^4-y^4=(x^2+y^2)(x^2-y^2)=(x^2+y^2)(x+y)(x-y)$ and $\frac{(x^2+y^2)(x+y)(x-y)}{x-y}=(x^2+y^2)(x+y)=x^3+x^2y+xy^2+y^3$

$$
\begin{array}{l}
(11.)\ 6x-7)\ 18x^3-33x^2+44x-35\ (3x^2-2x+5 \\
\qquad\qquad 18x^3-21x^2 \\
\hline
\qquad\qquad\quad -12x^2+44x \\
\qquad\qquad\quad -12x^2+14x \\
\hline
\qquad\qquad\qquad 30x-35 \\
\qquad\qquad\qquad 30x-35 \\
\hline
\end{array}
$$

$$
\begin{array}{l}
1+x)\ 1-x\ (1-2x+2x^2-2x^3+\&c. \\
\qquad 1+x \\
\hline
\qquad -2x \\
\qquad -2x-2x^2 \\
\hline
\qquad\quad 2x^2 \\
\qquad\quad 2x^2+2x^3 \\
\hline
\qquad\qquad -2x^3 \\
\qquad\qquad -2x^3-2x^4 \\
\hline
\qquad\qquad\quad 2x^4,\ \&c.
\end{array}
$$

NOTE.—It is plain, that in the second of these examples the quotient would never terminate; and we clearly see, without carrying on the division any further, that the terms of the quotient would follow one another, to any extent, according to this law; namely,

$$\frac{1-x}{1+x}=1-2x+2x^2-2x^3+2x^4-2x^5+2x^6-2x^7+, \text{ \&c.}$$

As in arithmetic, so in algebra; at whatever remainder we stop the operation of division, that remainder, with the divisor written underneath, must be appended to the quotient in order to render that quotient complete. In the above example, the complete quotient may be written in any of the following forms: namely,

$$\frac{1-x}{1+x}=1-\frac{2x}{1+x}, \text{ or}=1-2x+\frac{2x^2}{1+x}, \text{ or}=1-2x+2x^2-\frac{2x^3}{1+x}, \text{ \&c., \&c.}$$

(12.) The first three examples here are very easily worked by aid of the theorems at page 9 (Algebra) already referred to. The numerator of the first fraction is the square of the denominator (Theorem 1); in other words, the dividend is the square of the divisor. It is the same with the second fraction; and (by theorem 3) the numerator of the third fraction is $(2a^4+1)(2a^4-1)$. Hence, the three expressions are,

$$\frac{(a^2+b^2)(a^2+b^2)}{a^2+b^2}, \quad \frac{(x^3-y^3)(x^3-y^3)}{x^3-y^3}, \quad \frac{(2a^4+1)(2a^4-1)}{2a^4+1};$$

consequently, the results are a^2+b^2, x^3-y^3, and $2a^4-1$.

To find the result in the fourth example, we have only to subtract the exponent $m-2$ from the several exponents of a in the numerator or dividend, and to subtract the exponent n from the several exponents of b, leaving the dividend in other respects untouched: the result is, therefore,

$$a^2b^{n+2}-a^3b^{n+1}+a^4b^n.$$

In the first term of this, the exponent 2 is put over a, and the expouent $n+2$ over b, because $m-(m-2)=2$, and $2n+2-n=n+2$; in the second term, the exponent 3 is put over

a, and the exponent $n+1$ over b, because $m+1-(m-2)=3$, and $2n+1-n=n+1$; and lastly, in the third term, the exponent 4 is put over a, and the exponent n over b, because $m+2-(m-2)=4$, and $2n-n=n$. The learner must always keep in mind, that when a quantity with an exponent over it is to be multiplied by *the same quantity* with an exponent over it, the product is expressed by simply writing down that quantity with the *sum* of the exponents over it; and that when a quantity with an exponent over it is to be divided by the same quantity with an exponent over it, the quotient is expressed by simply writing down the quantity with the *difference* of the exponents over it—this difference arising from subtracting the exponent in the divisor from that in the dividend, as in the example above.

Miscellaneous Examples (Page 12).

(1.) Putting the proposed numbers, instead of the letters which stand for them, we have

$$4^2+12-18=16+12-18=10.$$

$$(2.)\ \frac{8+18}{13}-\frac{6}{3}=\frac{26}{13}-\frac{6}{3}=0.$$

$$(3.)\ \frac{4(6-3)}{3}+\frac{6}{4-3}=4+6=10.$$

(2.)
$$\begin{array}{l}
5a^2b-2ab^2-3a^2b^2\\
-7a^2b+5ab\ +2a^2b^2\\
-\ a^2b+\ ab^2\\
\hline
-3a^2b-ab^2-a^2b^2+5ab\\
\hline
\end{array}$$

(3.)
$$\begin{array}{l}
5x^3-\ x^2-\ 2x+1\\
2x^3-3x^2+\ 8x-5\\
\hline
3x^3+2x^2-10x+6\\
\hline
\end{array}$$

(4.)
$$\begin{array}{l}
4a^3x^2-\ 7a^2x\ -3a\\
2ax^3-\quad a^2\\
\hline
8a^4x^5-14a^3x^4-6a^2x^3\\
\qquad -4a^5x^2+7a^4x\ +3a^3\\
\hline
8a^4x^5-14a^3x^4-6a^2x^3-4a^5x^2+7a^4x+3a^3\\
\hline
\end{array}$$

$$\begin{array}{l} a^2+ab+b^2 \\ a-b \\ \hline a^3+a^2b+ab^2 \\ \quad -a^2b-ab^2-b^3 \\ \hline a^3 \qquad\qquad -b^3 \\ \hline \end{array}$$

(5.) $x^2+xy+y^2)\, x^4+x^2y^2+y^4\, (x^2-xy+y^2$

$$\begin{array}{l} x^4+x^3y+x^2y^2 \\ \hline -x^3y+y^4 \\ -x^3y-x^2y^2-xy^3 \\ \hline x^2y^2+xy^3+y^4 \\ x^2y^2+xy^3+y^4 \\ \hline \end{array}$$

$x+y)\, x^3+y^3\, (x^2-xy+y^2$ *

$$\begin{array}{l} x^3+x^2y \\ \hline -x^2y+y^3 \\ -x^2y-xy^2 \\ \hline xy^2+y^3 \\ xy^2+y^3 \\ \hline \end{array}$$

(6.)
$$\begin{array}{r} a\sqrt[3]{x}-\ b\sqrt{x} \\ 5a\sqrt[3]{x}+3b\sqrt{x} \\ -7a\sqrt[3]{x}+2b\sqrt{x} \\ 3a\sqrt[3]{x}+4b\sqrt{x} \\ \hline 2a\sqrt[3]{x}+8b\sqrt{x} \\ \hline \end{array}$$

* The learner will notice from this result, that the sum of two cubes is divisible by the sum of the quantities cubed; the form of the quotient, too—namely, the sum of the squares of those quantities, *minus* their product—should be kept in remembrance.

If in the above example y be made negative, then we shall have $\frac{x^3-y^3}{x-y}=x^2+xy+y^2$, which should also be recollected. These two theorems, like those at p. 9 of the Algebra, will frequently be referred to in the article on Fractions.

(6.) $(3a-\ x)\,x$ or $3ax-\ x^2$

$\ (\ a+2x)\,x \qquad ax+2x^2$

$\ (2a-3x)\,x \qquad 2ax-3x^2=(2a-3x)\,x.$

(7.) The theorems at page 9 (Algebra) readily enable us to effect the separation here required: thus,

$a^2-x^2=(a+x)(a-x)$; $a^2+2ax+x^2=(a+x)\ a+x)$;

$x^4-1=(x^2+1)\ (x^2-1)=(x^2+1)(x+1)(x-1)$; $4a^4-9b^2=(2a^2+3b)(2a^2-3b)$;

$x^2-4bx+4b^2=(x-2b)(x-2b)$; $a^{2m}-b^{2n}=(a^m+b^n)(a^m-b^n)$.

$$
\begin{array}{l}
ax-b)\ ax^3-(a^2+b)\,x^2+b^2\ (x^2-ax-b\,;\\
ax^3\quad\ -bx^2\\
\hline
-a^2x^2+b^2\\
-a^2x^2+abx\\
\hline
-abx+b^2\\
-abx+b^2\\
\hline
\end{array}
$$

(8.)

$$
\begin{array}{l}
px^2+qx-r\\
mx-n\\
\hline
pmx^3+qmx^2-mrx\\
\quad -pnx^2-qnx+nr\\
\hline
pmx^3+(qm-pn)x^2-(mr+qn)x+nr.\\
\hline
\end{array}
$$

(9.) aaa; $m(n-1$; $a(b^2-1)=a(b+1)\,(b-1)$; $(bc-b^2-b)x^2=b(c-b-1)xx$; $-(c^2+b+1)x$.

(10.) m^2n+mn^2; x^2y-xy^2; p^2-r^2; $-rx-avx$; $-\{x^2+x-1\}y=-x^2y+xy-y=xy-x^2y-y$.

Simple Equations (Page 13).

(1.) $x+3=18-4x$.

Transposing the $-4x$ to the left, and the $+3$ to the right, we have

$$x+4x=18-3;$$

that is, $5x=15$

$\therefore$ dividing by 5, $x=3$.

(2.) $x+3a=18a-4x$.

Transposing the $-4x$ to the left, and the $+3a$ to the right, we have

$$x+4x=18a-3a;$$

that is, $5x=15a$

$\therefore$ dividing by 5, $x=3a$.

(3.) $4x-2a=3x+2b$.

Transposing the $3x$ to the left, and the $-2a$ to the right, we have

$$4x-3x=2a+2b;$$

that is, $x=2(a+b$.

(4.) $7+6x-4=12+3x$.

Transposing the $+3x$ to the left, and the known numbers to the right, we have

$$6x-3x=12+4-7;$$

that is, $3x=9$

$\therefore$ dividing by 3, $x=3$.

(5.) $4(x-2)=10x-38$.

Removing the vinculum,

$$4x-8=10x-38.$$

Transposing the $4x$ to the right, and the -38 to the left, we have

$$38-8=10x-4x;$$

that is, $30=6x$

$\therefore$ dividing by 6, $5=x$.

Note.—It is usual to bring the unknown terms to the left, and the known to the right; but when such an arrangement would render the unknown side *minus*, as here, it is better to depart from it.

(6.) $ax+c=a-bx$.

Transposing, $ax+bx=a-c$,

or $(a+b)x=a-c$

$\therefore$ dividing by $a+b$

$$x=\frac{a-c}{a+b}.$$

$$(7.)\ \frac{y}{12}-8=-6.$$

Multiplying by 12, $y-96=-72$.

Transposing, $y=96-72=24$.

$$(8.)\ 5az-1=3a(z+b).$$

Removing the vinculum, $5az-1=3az+3ab$.

Transposing, $5az-3az=3ab+1$; that is, $2az=3ab+1$

$\therefore$ dividing by $2a$, $z=\dfrac{3ab+1}{2a}$.

Problems.

(3.) Suppose A has x pounds; then, by the question B must have $2x$ pounds, and C, $x+2x$ or $3x$ pounds; and these three shares make up the whole £600: hence we have the following equation—namely,

$$x+2x+3x=600;$$

that is, $6x=600 \therefore x=$ £100, A's share,

$\therefore 2x=$ £200, B's share, and $3x=$ £300, C's share.

(4.) Let x represent the less of the two numbers, then the greater must be $x+7$; and the question, moreover, informs us that

$$3(x+7)-8x=6.$$

Removing the vinculum, $3x+21-8x=6$.

Transposing, $21-6=8x-3x$

$\therefore 15=5x \therefore 3=x$:

Hence the two numbers are 3 and 10.

(5.) Let x represent the son's share; then $5x$ must represent the father's share; so that by the question we must have

$5x+x=96$; that is, $6x=96 \therefore x=16 \therefore 5x=80$

Hence the son's share was 16*s.*, and the father's 80*s.* or £4.

(6.) Suppose the investment was x pounds: then, after the gain and loss, A has $x+$ £300, and B, $x-$ £450; and the question tells us that the former sum is six times the latter; therefore,

$x+300=6x-2700$. Transposing, $2700+300=5x$,

that is, $3000=5x$;

therefore, dividing by 5, $600=x$, so that the investment was £600.

(7.) Let x feet be the length of the less part; then by the question bx is the length of the greater part; and since the sum of the parts is a, we have the equation,

$$x+bx=a;\text{ that is, }(1+b)x=a.\therefore x=\frac{a}{1+b},$$

$\therefore bx=\frac{ab}{1+b}$: consequently, the parts are $\frac{a}{1+b}$ feet and $\frac{ab}{1+b}$ feet.

If the whole length a be 20, and the greater part be 4 times the less, then since $a=20$ and $b=4$, we have

$$\frac{a}{1+b}=\frac{20}{5}=4,\text{ and }\frac{ab}{1+b}=\frac{80}{5}=16.$$

FRACTIONS (Page 16).

(2.) $\frac{x}{3}\times 24=8x$; for this is the same as $\frac{x}{3}\times 3\times 8=x\times 8$.

$\frac{3x}{4}\times 24=18x$; .... $\frac{3x}{4}\times 4\times 6=3x\times 6$.

$-\frac{5x}{6}\times 24=-20x$; .... $-\frac{5x}{6}\times 6\times 4=-5x\times 4$.

$\frac{3x}{8}\times 24 \quad =9x$; .... $\frac{3x}{8}\times 8\times 3=3x\times 3$.

$-\frac{7x}{12}\times 24=-14x$; .... $-\frac{7x}{12}\times 12\times 2=-7x\times 2$.

Sum of results $= \quad x$.

(3.) The denominators are all factors of the multiplier 20, hence we have only to expunge each denominator, and to multiply the numerator by the remaining factor of 20, after having expunged that factor which is equal to the denominator: we thus get

$$30y-8y-5y+14y-25y=6y.$$

(4.) $\left(\frac{4}{x}-\frac{7}{2x}+\frac{3}{8x}-\frac{9}{4x}+\frac{3}{2}\right)8x=32-28+3-18+12x=$ $12x-11$.

(5.) $\left(\frac{x}{5}+\frac{5}{x}-\frac{x}{10}-\frac{2}{x^2}+3\right)10x^2=2x^3+50x-x^3-20+30x^2$ $=x^3+30x^2+50x-20$.

Greatest Common Measure and Least Common Multiple (Page 21).

(1.)
$$
\begin{array}{l}
x^2+x-2)\ x^2+2x-3\ (1 \\
\qquad\qquad x^2+\ x-2 \\
\hline
\text{G. C. M.}=x-1)\ x^2+x-2\ (x+2 \\
\qquad\qquad x^2-x \\
\hline
\qquad\qquad 2x-2 \\
\qquad\qquad 2x-2
\end{array}
$$

or thus: $x^2+x-2=(x^2-1)+(x-1)$
$=(x-1)(x+1)+x-1$
$x^2+2x-3=(x^2-1)+2(x-1)$
$=(x-1)(x+1)+2(x-1)$
$\therefore$ G. C. M. $=x-1$.

(2.)
$$
\begin{array}{l}
6a^2+7ax-3x^2)\ 6a^2+11ax+3x^2\ (1 \\
\qquad\qquad 6a^2+\ 7ax-3x^2 \\
\hline
\qquad\qquad 4ax+6x^2; \\
\text{or G. C. M.}=2a+3x)\ 6a^2+7ax-3x^2\ (3a-x \\
\qquad\qquad 6a^2+9ax \\
\hline
\qquad\qquad -2ax-3x^2 \\
\qquad\qquad -2ax-3x^2.
\end{array}
$$

(3.) In order to remove common factors, we shall first divide each of the given expressions by b, remembering, however, that the b thus removed, being a common measure of both, must necessarily enter into the *greatest* common measure. The two expressions will then be $8a^2b-10ab^2+2b^3$, and $9a^4-9a^3b+3a^2b^2-3ab^3$. We shall next divide each term of the first of these expressions by $2b$, and each term of the

second by $3a$; for in so doing, we shall not interfere with any *common* measure the two expressions may have, seeing that neither 3 nor a is a factor of the first expression, nor 2 nor b of the second: we shall then have to deal with $4a^2-5ab+b^2$, and $3a^3-3a^2b+ab^2-b^3$. Lastly, we shall reverse the two expressions to avoid fractions in the quotient; the work will then stand as follows:—

$$
\begin{array}{l}
b^2-5ab+4a^2)-b^3+ab^2-3a^2b+3a^3(-b-4a \\
\qquad\qquad\qquad -b^3+5ab^2-4a^2b \\
\hline
\qquad\qquad\qquad\qquad -4ab^2+a^2b+3a^3 \\
\qquad\qquad\qquad\qquad -4ab^2+20a^2b-16a^3 \\
\hline
\qquad\qquad\qquad\qquad\qquad -19a^2b+19a^3;
\end{array}
$$

$$
\begin{array}{l}
\text{or } -b+a)b^2-5ab+4a^2(-b+4a \\
\qquad\qquad b^2-\ ab \\
\hline
\qquad\qquad\quad -4ab+4a^2 \\
\qquad\qquad\quad -4ab+4a^2. \\
\hline
\end{array}
$$

The greatest common measure here is $-b+a$; but as the common factor, b, was suppressed at the outset, we have for the greatest common measure of the proposed expression, G. C. M. $=-b^2+ab$, or $ab-b^2$.

$$
\begin{array}{l}
(4.)\ x^3-2x-1)x^3+2x^2+2x+1(1 \\
\qquad\qquad\qquad\quad x^3-2x-1 \\
\hline
\qquad\qquad\qquad\qquad 2x^2+4x+2;
\end{array}
$$

$$
\begin{array}{l}
\text{or } x^2+2x+1)x^3-2x-1(x-2 \\
\qquad\qquad\quad x^3+2x^2+x \\
\hline
\qquad\qquad\quad -2x^2-3x-1 \\
\qquad\qquad\quad -2x^2-4x-2 \\
\hline
\end{array}
$$

G. C. M. $=x+1)x^2+2x+1(x+1$. (Theorem 1, p. 9, Algebra.)

(3.) To avoid fractions, multiply the first expression by 3, and the second by 2; the work will then stand as follows:—

$$
\begin{array}{l}
6x^2-3xy-18y^2)6x^2-16xy+8y^2(1 \\
\qquad\qquad\qquad\quad 6x^2-3xy-18y^2 \\
\hline
\qquad\qquad\qquad\qquad -13xy+26y^2;
\end{array}
$$

$$\text{or G. C. M.}=x-2y)2x^2-xy-6y^2(2x+3y$$
$$\begin{array}{l} 2x^2-4xy \\ \hline \quad\quad 3xy-6y^2 \\ \quad\quad 3xy-6y^2 \\ \hline \end{array}$$

(6.) Instead of proceeding by actual division, we may find the common measure more readily as follows:—

$$2x^3-2x-3x^2+3=2x(x^2-1)-3(x^2-1)=(2x-3)(x^2-1)$$
$$3x^4-3+2x^3-2x-2x^2+2=3(x^4-1)+2x(x^2-1)-2(x^2-1).$$

The expressions on the right are the same virtually as those in the book; and it is plain that the G. C. M. of these is x^2-1.

$$(7.)\ \frac{x^2+2ax+a^2}{x^2-a^2}=\frac{(x+a)(x+a)}{(x+a)(x-a)}=\frac{x+a}{x-a};$$
$$\frac{x^2-2xy+y^2}{x^2-y^2}=\frac{(x-y)(x-y)}{(x+y)(x-y)}=\frac{x-y}{x+y}.$$

(8.) See foot-note, p. 10.

$$\frac{a^3+b^3}{(a+b)^2}=\frac{(a+b)(a^2-ab+b^2)}{(a+b)(a+b)}=\frac{a^2-ab+b^2}{a+b};$$
$$\frac{(a+b)^3}{a^2-b^2}=\frac{(a+b)(a+b)^2}{(a+b)(a-b)}=\frac{(a+b)^2}{a-b}.$$

$$(9.)\ \frac{x^2+x-2}{2x^2-3x+1}=\frac{x^2-1+x-1}{2x^2-2-3x+3}=\frac{(x+1)(x-1)+x-1}{2(x+1)(x-1)-3(x-1)}$$
$$=\frac{x+1+1}{2(x+1)-3}=\frac{x+2}{2x-1}$$

$$\frac{x^{3m}+x^{2m}-2}{x^{2m}+x^m-2}=\frac{x^{3m}-1+x^{2m}-1}{x^{2m}-1+x^m-1}\,^*=$$
$$\frac{(x^m-1)(x^{2m}+x^m+1)+(x^m-1)(x^m+1)}{(x^m-1)(x^m+1)+x^m-1}=$$
$$\frac{x^{2m}+x^m+1+x^m+1}{x^m+1+1}=\frac{x^{2m}+2x^m+2}{x^m+2};$$

or, by finding the common measure of numerator and denominator in the ordinary way, we have

* See the foot-note at page 10.

$$
\begin{array}{l}
x^{2m}+x^m-2)x^{3m}+x^{2m}-2(x^m \\
\qquad\qquad\quad x^{3m}+x^{2m}-2x^m \\
\hline
\qquad\qquad\qquad\quad 2x^m-2;
\end{array}
$$

$$
\begin{array}{l}
\text{or G. C. M.}=x^m-1)x^{2m}+x^m-2(x^m+2 \\
\qquad\qquad\qquad\quad x^{2m}-x^m \\
\hline
\qquad\qquad\qquad\qquad\quad 2x^m-2 \\
\qquad\qquad\qquad\qquad\quad 2x^m-2 \\
\hline
\end{array}
$$

$$
\begin{array}{l}
x^m-1)x^{3m}+x^{2m}-2(x^{2m}+2x^m+2 \\
\qquad\quad x^{3m}-x^{2m} \\
\hline
\qquad\qquad 2x^{2m}-2 \\
\qquad\qquad 2x^{2m}-2x^m \\
\hline
\qquad\qquad\qquad 2x^m-2 \\
\qquad\qquad\qquad 2x^m-2 \\
\hline
\end{array}
$$

The common measure being thus found, the proposed fraction becomes reduced to the simpler form arrived at above.

(10.) The numerator of the first of these fractions may be written thus:

$x(x^2-y^2)-y(x^2-y^2)$, or $(x-y)(x^2-y^2)$; hence the fraction is

$$\frac{(x-y)(x^2-y^2)}{(x^2+y^2)(x^2-y^2)}=\frac{x-y}{x^2+y^2}.$$

The factor x, common to all the terms of the denominator of the second fraction, but not common to those of the numerator, may be suppressed, since no *common measure* of the two expressions will be interfered with by so doing: we shall therefore have to find the greatest common measure of

$$3x^3-22x-15 \text{ and } 5x^3-17x^2+18.$$

If we proceed by the ordinary method (as at p. 21, Algebra), the work will become tedious: we may dispense with it as follows:—

The leading term of the common divisor of these expressions, if they have a common divisor, cannot have a coefficient other than unity, because 3 and 5 have no common divisor.

The leading term of such divisor must, therefore, be simply

x; it cannot be x^2, because, on account of the simple power of x (or the term $22x$) entering the first expression, that expression cannot be divided by any other in which the x, simply, is absent, as is obvious.

The final term of the divisor must, of course, divide both 15 and 18, and therefore can be no other than 3: hence, if the expressions have a common divisor, that divisor must be either $x-3$ or $x+3$. Upon trial, the former is found to succeed, and the latter to fail: hence the greatest common measure is $x-3$: thus

$$\begin{array}{l} x-3)3x^3-22x-15(3x^2+9x+5 \\ \quad\;\; 3x^3-\;9x^2 \\ \hline \qquad\qquad 9x^2-22x \\ \qquad\qquad 9x^2-27x \\ \hline \qquad\qquad\qquad 5x-15 \\ \qquad\qquad\qquad 5x-15 \\ \hline \end{array}$$

$$\begin{array}{l} x-3)5x^4-17x^3+18x(5x^3-2x^2-6x \\ \quad\;\; 5x^4-15x^3 \\ \hline \qquad\; -2x^3+18x \\ \qquad\; -2x^3+\;6x^2 \\ \hline \qquad\qquad -\;6x^2+18x \\ \qquad\qquad -\;6x^2+18x \\ \hline \end{array}$$

Consequently, the fraction in its lowest terms is $\dfrac{3x^2+2x+5}{5x^3-2x^2-6x}$

NOTE.—The operation for the common measure, which is sometimes long and troublesome, may frequently be dispensed with by the help of considerations such as those here pointed out: for instance, in example 1 (p. 21, Algebra), it is plain that the leading term of the divisor common to both expressions must be x; and since 2 and 3 have no common divisor besides unit, it follows that the greatest common measure, if such exist, must be either $x-1$ or $x+1$. For similar reasons, the G. C. M. in example 4 must be either $x-1$ or $x+1$, and the same in the first example of 9.

Addition of Fractions (Page 26).

(1.) The least common multiple of the denominators is evidently 12; so that multiplying the terms of the first fraction by 6, those of the second by 4, and those of the third by 3, the fractions become changed into

$$\frac{6x}{12}+\frac{8x}{12}+\frac{9x}{12}=\frac{23x}{12}=x+\frac{11x}{12}.$$

(2.) The L. C. M. of the denominators is 20, so that the terms of the first two fractions are to be multiplied by 2, 5 and those of the third to remain untouched: we thus have

$$\frac{4x}{10}+\frac{15x}{10}+\frac{7x}{10}=\frac{26x}{10}=\frac{13x}{5}$$

$$\frac{x}{a+x}+\frac{a}{a-x}=\frac{x(a-x)+a(a+x)}{(a+x)(a-x)}=\frac{a^2+2ax-x^2}{a^2-x^2}.$$

(3.) $$\frac{3a^2}{2b}+\frac{2a}{5}+\frac{3b}{7a}=\frac{3a^2\times 35a+2a\times 14ab+3b\times 10b}{70ab}=$$

$$\frac{105a^3+28a^2b+30b^2}{70ab};$$

$$\frac{x}{x+3}+\frac{x}{x-3}=\frac{x(x-3)+x(x+3)}{(x+3)(x-3)}=\frac{2x^2}{x^2-9}.$$

(4.) $$\frac{a+b}{a-b}+\frac{a-b}{a+b}=\frac{(a+b)^2+(a-b)^2}{a^2-b^2}=\frac{2(a^2+b^2)}{a^2-b^2};$$

$$\frac{2x}{1-x^2}+\frac{1}{1+x}=\frac{2x}{1-x^2}+\frac{1-x}{1-x^2}=\frac{1+x}{1-x^2}=\frac{1}{1-x}.$$

(5.) $$\frac{a^2+ab+b^2}{a+b}+\frac{b^2}{a-b}=\frac{(a-b)(a^2+ab+b^2)+b^2(a+b)}{a^2-b^2}$$

$$=\frac{a^3+a^2b+ab^2-a^2b-b^2(a+b)+b^2(a+b)}{a^2-b^2}=\frac{a(a^2+b^2)}{a^2-b^2}.$$

Or, remembering that $(a^2+ab+b^2)(a-b)=a^3-b^3$ (see footnote, p. 10), we have $\dfrac{a^3-b^3+ab^2+b^3}{a^2-b^2}=\dfrac{a(a^2+b^2)}{a^2-b^2}$.

$$\frac{x-1}{x^2+x+1}+\frac{1}{x-1}=\frac{(x-1)^2+x^2+x+1}{x^3-1}=\frac{2x^2-x+2}{x^3-1}.$$

(6.) $$\frac{x}{x^2-y^2}+\frac{y}{x+y}+\frac{1}{x-y}=\frac{x+y(x-y)+x+y}{x^2-y^2}=$$
$$\frac{2x+xy-y^2+y}{x^2-y^2};$$

$$\frac{a}{1-a}+\frac{1}{1+a}+\frac{a}{1+a}=\frac{a}{1-a}+\frac{1+a}{1+a}=\frac{a}{1-a}+1=\frac{a+1-a}{1-a}=$$
$$\frac{1}{1-a}.$$

Subtraction of Fractions (Page 27).

(1.) $$\frac{5a}{2}-\frac{a}{3}=\frac{15a-2a}{6}=\frac{13}{6}a;\ \frac{3(x+y)}{4}-\frac{x-y}{8}=$$
$$\frac{6(x+y)-x+y}{8}=\frac{5x+7y}{8}.$$

(2.) $$\frac{5y+2}{7}-\frac{2y+1}{3}=\frac{3(5y+2)-7(2y+1)}{21}=\frac{y-1}{21};$$

$$\frac{5x-3}{x+1}-\frac{3x+2}{x-1}=\frac{(x-1)(5x-3)-(x+1)(3x+2)}{x^2-1}=$$
$$\frac{5x^2-8x+3-3x^2-5x-2}{x^2-1}=\frac{2x^2-13x+1}{x^2-1}.$$

(3.) $$\frac{a+2b}{a-2b}-\frac{a-2b}{a+2b}=\frac{(a+2b)^2-(a-2b)^2}{a^2-4b^2}=\frac{8ab}{a^2-4b^2}.$$

NOTE.—It is well for the learner to remember, that the square of the sum diminished by the square of the difference of two quantities, is always equal to *four times* the product of those quantities. Thus $(x+y)^2-(x-y)^2=4xy$, $(a+2b)^2-(a-2b^2)=8ab$, &c. The results in such cases should always, therefore, be written down at once, without any actual work.

(4.) $$\frac{1}{y-z}-\frac{1}{y^2-z^2}=\frac{y+z-1}{y^2+z^2};\ \frac{2x^2-2x+1}{x^2-x}-\frac{x}{x-1}=$$
$$\frac{2x^2-2x+1-x^2}{x^2-x}=\frac{x^2-2x+1}{x(x-1)}=\frac{(x-1)^2}{x(x-1)}=\frac{x-1}{x}=1-\frac{1}{x}.$$

(5.) $\dfrac{x^2-x+1}{x-1}-\dfrac{2}{x+1}=\dfrac{(x+1)(x^2-x+1)-2(x-1)}{x^2-1}=$

$$\frac{x^3+1-2x+2}{x^2-1}=\frac{x^3-2x+3}{x^2-1};$$

$$\frac{a}{(1-a)^2}-\frac{a^2}{(1-a)^3}+\frac{1}{1-a}=\frac{a(1-a)-a^2+(1-a)^2}{(1-a)^3}=$$

$$\frac{a-a^2-a^2+1-2a+a^2}{(1-a)^3}=\frac{1-a-a^2}{(1-a)^3}.$$

(6.) $\dfrac{x^2+y^2}{x^2-y^2}+\dfrac{x}{x+y}-\dfrac{y}{x-y}=\dfrac{x^2+y^2+x(x-y)-y(x+y)}{x^2-y^2}=$

$$\frac{x^2+y^2+x^2-xy-xy-y^2}{x^2-y^2}=\frac{2x^2-2xy}{x^2-y^2}=\frac{2x(x-y)}{(x+y)(x+y)}=\frac{2x}{x+y}.$$

Also, $\dfrac{1}{(x-y)^2}-\dfrac{1}{x^2-y^2}=\dfrac{1}{(x-y)(x-y)}-\dfrac{1}{(x-y)(x+y)}$

$$=\frac{x+y-(x-y)}{(x+y)(x-y)^2}=\frac{2y}{(x+y)(x-y)^2}.$$

(7.) $\frac{1}{2}\left\{\dfrac{3m+2n}{3m-2n}-\dfrac{3m-2n}{3m+2n}\right\}=\frac{1}{2}\left\{\dfrac{(3m+2n)^2-(3m-2n)^2}{9m^2-4n^2}\right\}$

$$=\frac{1}{2}\,\frac{12mn}{9m^2-4n^2}=\frac{6mn}{9m^2-4n^2}.$$ (See Note above).

(8.) $\dfrac{x^{3n}}{x^n-1}-\dfrac{x^{2n}}{x^n+1}-\dfrac{1}{x^n-1}+\dfrac{1}{x^n+1}$. This may be written $\dfrac{x^{3n}-1}{x^n-1}-\dfrac{x^{2n}-1}{x^n+1}$, the first and third, as also the second and fourth fractions, being united in one. The numerator of the first of these is the difference of the cubes of x^n and 1: hence (foot-note, p.) it is actually divisible by the denominator, the quotient being $x^{2n}+x^n+1$. Also, the numerator of the second fraction, being the difference of the squares of x^n and 1, is divisible by the denominator: the quotient is x^n-1. Hence the sum of the fractions is $x^{2n}+x^n+1-(x^n-1)=x^{2n}+2$.

Multiplication of Fractions (Page 28).

(3.) $\dfrac{3a}{5}\times\dfrac{a}{4}=\dfrac{3a^2}{20}$; $\dfrac{2x}{3}\times\dfrac{xy^2}{6}=\dfrac{x}{3}\times\dfrac{xy^2}{3}=\dfrac{x^2y^2}{9}$.

NOTE.—Factors which enter alike into numerators and

denominators may be suppressed, as in common arithmetic, as it is useless to multiply and divide by the same thing: the operations mutually destroy one another, and therefore should both be omitted. The example here worked is $\frac{2x}{3}\times\frac{xy^2}{2\times3}$; and the factor 2, being common to a numerator and a denominator—that is, occurring both as a multiplier and divisor—is wholly expunged or neglected, and the example treated as simply $\frac{x}{3}\times\frac{xy^2}{3}$. The learner should always take advantage of these means of simplification, so as to avoid all unnecessary work. The next example may be simplified in like manner.

$$(2.)\ \frac{2x}{x-y}\times\frac{(x-y)(x-y)}{2\times4}=\frac{x(x+y)}{4}=\frac{x^2+xy}{4};$$

$$\frac{(x+1)(x-1)}{3}\times\frac{3\times2a}{x+1}=2a(x-1).$$

$$(3.)\ \frac{(a+b)(a-b)}{a}\times\frac{1}{a+b}\times\frac{a}{a-b}=1;\ \frac{3(5z-10)}{2z}\times\frac{3z^2}{5z-10}=\frac{9z}{2}.$$

NOTE.—In the first of these examples, the two factors $(a+b)(a-b)$ in the numerator are rendered of no effect by the equal factors in the following denominators; in like manner, the denominator a is neutralised by the last numerator. It will, of course, be observed, that when any factor is thus expunged, it is not replaced by 0, but by 1. In the present case, after suppressing the common factors, each remaining factor is merely 1, and the product of the fractions is therefore $\frac{1}{1}$, or 1.

$$(4.)\ \frac{(x-1)^2}{y^2}\times\frac{(x+1)y^2}{x-1}=\frac{x-1}{y}\times\frac{x+1}{1}=\frac{x^2-1}{y}.$$

$\left(m+\frac{1}{m}-1\right)\left(m+\frac{1}{m}+1\right)$. This is the sum of two quantities—namely, $\left(m+\frac{1}{m}\right)$ and 1, multiplied by their difference: we know, therefore, that the product is the difference of the squares of those quantities (Algebra, p. 9): hence the product is

$$\left(m+\frac{1}{m}\right)^2-1=m^2+2+\frac{1}{m^2}-1=m^2+1+\frac{1}{m^2}.$$

(5.) $\left(x-\frac{y^2}{x}\right)\left(\frac{x}{y}+\frac{y}{x}\right)$. This is evidently the same as $y\left(\frac{x}{y}-\frac{y}{x}\right)\left(\frac{x}{y}+\frac{y}{x}\right)=$ (Alg. p. 9), $y\left(\frac{x^2}{y^2}-\frac{y^2}{x^2}\right)=y\left(\frac{x^4-y^4}{x^2y^2}\right)=$

$$\frac{x^4-y^4}{x^2y}.$$

$$\frac{4x^4-1}{9x^6-y^2}.\quad \frac{3x^3+y}{2x^2-1}=\frac{(2x^2+1)(2x^2-1)}{(3x^3+y)(3x^3-y)}.\quad \frac{3x^3+y}{2x^2-1}=\frac{2x^2+1}{3x^3-y}$$

(6.) $\left\{\frac{a^4x}{a^2-x^2}-a^2x-x^3\right\}\times\frac{a+x}{x^3}=\left\{\frac{a^4}{a^2-x^2}-(a^2+x^2)\right\}\times$

$$\frac{a+x}{x^2}=\frac{a^4-(a^4-x^4)}{(a+x)(a-x)}\times\frac{a+x}{x^2}=\frac{x^4}{(a-x)x^2}=\frac{x^2}{a-x}.$$

Division of Fractions (Page 29).

(1.) $\frac{5m^2}{n}\times\frac{3n}{10}=\frac{5m^2}{n}\times\frac{3n}{5\times 2}=\frac{3m^2}{2}$; $\frac{3x}{2(x-1)}\times\frac{x-1}{2x}=\frac{3}{4}$.

(2.) $\frac{2(2a+1)}{3}\times\frac{5a}{2a+1}=\frac{10a}{3}$; $\frac{(x+y)(x+y)}{x-y}\times\frac{(x-y)(x-y)}{x+y}$

$$=\frac{(x+y)(x-y)}{1}=x^2-y^2.$$

(3.) $x+\frac{x}{x-1}=\frac{x^2}{x-1}$, and $x-\frac{x}{x-1}=\frac{x^2-2x}{x-1}$: hence the example is $\frac{x^2}{x-1}\times\frac{x-1}{x(x+2)}=\frac{x}{1}\times\frac{1}{x-2}=\frac{x}{x-2}$.

$\left(x^4-\frac{1}{x^2}\right)\div\left(x^2+\frac{1}{x}\right)$. This is the difference of the squares of two quantities divided by the sum of the quantities themselves; the quotient is, therefore, the difference of those quantities—namely, $x^2-\frac{1}{x}$ (Algebra, p. 9).

(4.) By actual division, $\frac{x^3-3x^2a+3xa^2-a^3}{x+a}=x^2-2xa+a^2$;

hence, $\frac{(x^3-3x^2a+3xa^2-a^3)(x-a)}{(x+a)(x-a)}=x^2-2xa+a^2=(x-a)^2.$

(5.) $\frac{(x^2+y^2)(x^2-y^2)}{a^3+b^3}\times\frac{a^2-ab+b^2}{x-y}$. Now, it has already been seen (p. 10), that $a^2-ab+b^2)(a+b)=a^3+b^3$: hence, suppressing the factor (a^2-ab+b^2), entering the first denominator and the second numerator, as also the factor $x-y$, in the first numerator and second denominator, we have for the product,

$$\frac{(x^2+y^2)(x+y)}{a+b}\times 1=\frac{x^3+x^2y+xy^2+y^3}{a+b}.$$

(6.) The first expression here is the same as $\left(x+\frac{1}{x}\right)^2$ (Algebra, p. 9). Hence the example is

$$\left(x+\frac{1}{x}\right)^2\times\frac{a}{x+\frac{1}{x}}=a\left(x+\frac{1}{x}\right)=a\frac{x^2+1}{x}=\frac{ax^2+a}{x}.$$

Problems (Page 37).

(1.) Suppose the present age of the son to be x years, then 10 years ago his age was $x-10$; and, consequently, the father's age then was ten times as much—namely, $10x-100$.

At present, the father's age, being four times the son's, is $4x$; so that 10 years ago his age was $4x-10$; but, as just seen, his age then was $10x-100$: hence, we must have the equation

$$10x-100=4x-10.$$

Transposing, $6x=90$ $\therefore$ $x=15$, the son's present age;

$\therefore$ $4x=60$, the father's present age.

(2.) Let x represent the number; then by the question,

$\frac{3x-8}{2}=x-2$ $\therefore$ $3x-8=2x-4$. Transposing, $x=4$.

(3.) Let $\frac{x}{y}$ represent the fraction: then by the question

$$\frac{x+1}{y}=\frac{1}{3}, \text{ and } \frac{x}{y-1}=\frac{1}{4}.$$

Clearing the fractions, these become

$$3x+3=y \text{ and } 4x=y-1 \therefore 4x+1=y.$$

Putting this value of y for that symbol in the first of these equations, $3x+3=4x+1$. Transposing, $2=x.\therefore y=4x+1=9$.

Hence the fraction is $\frac{2}{9}$.

(4.) Suppose A's share to be x pounds; then B's must be $\frac{4}{5}x$, and C's $\frac{5}{6}\left(x+\frac{4}{5}x\right)$, or $\frac{5}{6}$ of $\frac{9}{5}x$, that is, $\frac{3}{2}x$. Hence the sum of the shares is

$x+\frac{4}{5}x+\frac{3}{2}x=132$. Multiplying by 10, $10x+8x+15x=1320$;

that is, $33x=1320 \therefore x=40 \therefore \frac{4}{5}x=32$, and $\frac{3}{2}x=60$.

Hence, A's share is £40, B's £32, and C's £60.

Otherwise.—Suppose A's share to be $5x$ pounds; then B's must be $4x$, and C's $\frac{5}{6}(5x+4x)=\frac{5}{6}\cdot 9x=\frac{15}{2}x$.

$\therefore 5x+4x+\frac{15}{2}x=132$. Multiplying by 2, $10x+8x+15x=264$

$\therefore 33x=264 \therefore x=8 \therefore 5x=40$, $4x=32$, and $\frac{15}{2}x=60$.

Hence the shares are £40, £32, and £60.

(5.) Suppose he had x sovereigns; then, after giving $\frac{3}{10}$ to B, he had $\frac{7}{10}x$; and after giving $\frac{6}{7}$ of this to C, he had $\frac{1}{7}$ of $\frac{7}{10}x$ left—that is, $\frac{1}{10}x$. Consequently, whatever number x may represent, $\frac{1}{10}$ of that number will be left after the deductions stated in the question: hence, there is no limitation as to the number of A's sovereigns.

(6.) To avoid unnecessary fractions, suppose he had $12x$ pounds; then he spent in horses and oxen x and $4x$, so that $7x$ pounds were left; and after spending $\frac{3}{10}$ of this in sheep, $\frac{7}{10}$ of it was left—that is,

$$\frac{49x}{10} \therefore \frac{49x}{10} = 98 \therefore 49x = 980$$

$$\therefore x = 20 \therefore 12x = 240 \therefore \text{he had £240.}$$

Involution (Page 39).

The first *five* examples under this head should be written, without any actual work, from the theorems 1 and 2 (p. 9, Algebra).

(6.) $(a+b)^4 = (a+b)^2(a+b)^2 = (a^2+2ab+b^2)(a^2+2ab+b^2)$

$$\begin{array}{l} a^2+2ab+b^2 \\ a^2+2ab+b^2 \\ \hline a^4+2a^3b+a^2b^2 \\ \quad\ 2a^3b+4a^2b^2+2ab^3 \\ \qquad\qquad a^2b^2+2ab^3+b^4 \\ \hline a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\ \hline \end{array}$$

or thus: $\{(a^2+2ab)+b^2\}^2$ by theorem 1 (p. 9, Algebra).

$$\begin{aligned} &= (a^2+2ab)^2+2(a^2+2ab)b^2+b^4 \\ &= a^4+4a^3b+4a^2b^2+2a^2b^2+4ab^3+b^4 \\ &= a^4+4a^3b+6a^2b^2+4ab^3+b^4. \end{aligned}$$

(7.) $(a-x)^4 = \{(a^2-2ax)+x^2\}^2 = (a^2-2ax)^2+2(a^2-2ax)x^2+x^4$

$$\begin{aligned} &= a^4-4a^3x+4a^2x^2+2a^2x^2-4ax^3+x^4 \\ &= a^4-4a^3x+6a^2x^2-4ax^3+x^4. \end{aligned}$$

(8.) From ex. 6,

$(x-y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$; therefore, multiplying by $x+y$

$$\begin{array}{l} x^5+4x^4y+6x^3y^2+4x^2y^3+xy^4 \\ \qquad x^4y+4x^3y^2+6x^2y^3+4xy^4+y^5 \\ \hline (x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5. \\ \hline \end{array}$$

(9.) In the last example, change x into s, and y into z, and there results $(s-z)^5 = s^5-5s^4z+10s^3z^2-10s^2z^3+5sz^4-z^5$.

(10.) This is the same as $(\sqrt{x}+\sqrt{y})^2 = x+2\sqrt{xy}+y$.

(11.) This is a case of theorem 2 (p. 9, Algebra).

c 2

(12.) Powers of the same letter are multiplied together by merely writing over the letter the sum of the exponents;

$$\text{therefore, } (x^2y+y^{\frac12})^2=x^4y^2+2x^2y^1y^{\frac12}+y=$$
$$x^4y^2+2x^2y^{1+\frac12}+y=x^4y^2+2x^2y^{\frac32}+y$$

(13.) The cube is at once obtained by substituting $x^{\frac13}$, $y^{\frac13}$ for a, b in the expression for $(a-b)^3$ (p. 38, Algebra). Or the work may be executed as in the margin, by first writing down the square of $x^{\frac13}-y^{\frac13}$ by theorem 2 (p. 9, Algebra), and then multiplying by $x^{\frac13}-y^{\frac13}$.

$$\begin{array}{r}
(x^{\frac13}-y^{\frac13})^2=x^{\frac23}-2x^{\frac13}y^{\frac13}+y^{\frac23} \\
x^{\frac13}-y^{\frac13} \\
\hline
x-2x^{\frac23}y^{\frac13}+x^{\frac13}y^{\frac23} \\
-x^{\frac23}y^{\frac13}+2x^{\frac13}y^{\frac23}-y \\
\hline
x-3x^{\frac23}y^{\frac13}+3x^{\frac13}y^{\frac23}-y. \\
\hline
\end{array}$$

$$\begin{array}{r}
(14.)\quad (x+1)^2=x^2+2x+1 \\
x^2+2x+1 \\
\hline
x^4+2x^3+x^2 \\
2x^3+4x^2+2x \\
x^2+2x+1 \\
\hline
x^4+4x^3+6x^2+4x+1 \\
\hline
\end{array}$$

(15.) By the theorem at the top of p. 39, Algebra, the required cube is equal to the difference of the cubes of $(a+b)^{\frac13}$, and $(a-b)^{\frac13}$, *minus* three times their product multiplied by that difference. Now, the difference of the cubes is $(a+b)-(a-b)=2b$: hence, subtracting three times the product multiplied by the difference, we have

$$\{(a+b)_{\frac13}-(a-b)^{\frac13}\}^3=2b-3(a+b)^{\frac13}(a-b)^{\frac13}\{(a+b)^3-(a-b)^{\frac13}\}$$

This result is obtained from imitating the general model for the cube of the difference of *any* two quantities at p. 38, Algebra; but, as an exercise in actual involution, it may be well to exhibit the work at length, as follows:—

$$
\begin{array}{l}
(a+b)^{\frac{1}{3}}-(a-b)^{\frac{1}{3}} \\
\underline{(a+b)^{\frac{1}{3}}-(a-b)^{\frac{1}{3}}} \\
(a+b)^{\frac{2}{3}}-(a+b)^{\frac{1}{3}}(a-b)^{\frac{1}{3}} \\
\qquad \underline{-(a+b)^{\frac{1}{3}}(a-b)^{\frac{1}{3}}+(a-b)^{\frac{2}{3}}} \\
(a+b)^{\frac{2}{3}}-2(a+b)^{\frac{1}{3}}(a-b)^{\frac{1}{3}}+(a-b)^{\frac{2}{3}} \\
\underline{(a+b)^{\frac{1}{3}}-(a-b)^{\frac{1}{3}}} \\
(a+b)-2(a+b)^{\frac{2}{3}}(a-b)^{\frac{1}{3}}+(a+b)^{\frac{1}{3}}(a-b)^{\frac{2}{3}} \\
\quad \underline{-\ (a+b)^{\frac{2}{3}}(a-b)^{\frac{1}{3}}+2(a+b)^{\frac{1}{3}}(a-b)^{\frac{2}{3}}-(a-b)} \\
(a+b)-3(a+b)^{\frac{2}{3}}(a-b)^{\frac{1}{3}}+3(a+b)^{\frac{1}{3}}(a-b)^{\frac{2}{3}}-(a-b) \\
=2b-3(a+b)^{\frac{1}{3}}(a-b)^{\frac{1}{3}}\{(a+b)^{\frac{1}{3}}-(a-b)^{\frac{1}{3}}\}.
\end{array}
$$

Evolution (Page 45).

(6.)

$$
\begin{array}{rl}
 & 4x^2-4ax+a^2(2x-a \\
 & \underline{4x^2} \\
4x-a) & -4ax+a^2 \\
 & \underline{-4ax+a^2}
\end{array}
$$

(7.)

$$
\begin{array}{rl}
 & 1-8a+16a^2(1-4a \\
 & \underline{1} \\
2-4a) & -8a+16a^2 \\
 & \underline{-8a+16a^2}
\end{array}
$$

$$
\begin{array}{rl}
 & 25a^2+40ax+16x^2(5a+4x \\
 & \underline{25a^2} \\
10a+4x) & 40ax+16x^2 \\
 & \underline{40ax+16x^2}
\end{array}
$$

(8.)

$$
\begin{array}{rl}
 & 9x^2-3x+\frac{1}{4}(3x-\frac{1}{2} \\
 & \underline{9x^2} \\
6x-\frac{1}{2}) & -3x+\frac{1}{4} \\
 & \underline{-3x+\frac{1}{4}}
\end{array}
$$

(9.)

$$\begin{array}{r|l} & 49a^4+42a^2b+9b^2(7a^2+3b \\ & 49a^4 \\ \hline 14a^2+3b) & 42a^2b+9b^2 \\ & 42a^2b+9b^2 \\ \hline \end{array}$$

(10.)

$$\begin{array}{r|l} & x^2-2ax+a^2+2x-2a+1(x-a+1 \\ & x^2 \\ \hline 2x-a) & -2ax+a^2 \\ & -2ax+a^2 \\ \hline 2x-2a+1) & 2x-2a+1 \\ & 2x-2a+1 \\ \hline \end{array}$$

(11.)

$$\begin{array}{r|l} & m^2+2m-1-\frac{2}{m}+\frac{1}{m^2}\Big(m+1-\frac{1}{m} \\ & m^2 \\ \hline 2m+1) & 2m-1 \\ & 2m+1 \\ \hline 2m+2-\frac{1}{m}\Big) & -2-\frac{2}{m}+\frac{1}{m^2} \\ & -2-\frac{2}{m}+\frac{1}{m^2} \\ \hline \end{array}$$

(12.)

$$\begin{array}{r|l} & \frac{a^2}{x^2}-2+\frac{x^2}{a^2}+\frac{2a^3}{x}-2x+a^2\Big(\frac{a}{x}-\frac{x}{a}+a \\ & \frac{a^2}{x^2} \\ \hline \frac{2a}{x}-\frac{x}{a}\Big) & -2+\frac{x^2}{a^2} \\ & -2+\frac{x^2}{a^2} \\ \hline \frac{2a}{x}-\frac{2x}{a}+a\Big) & \frac{2a^3}{x}-2x+a^2 \\ & \frac{2a^3}{x}-2x+a^2 \\ \hline \end{array}$$

(13.) The common method of extracting the cube root, whether in arithmetic or in algebra, is difficult to remember. We shall replace it here by a simple and obvious operation, analogous to that exhibited for numbers in the "Rudimentary Arithmetic," a reference to which will enable the learner to comprehend what follows at a glance, as the two processes are the same.

$$\begin{array}{lll}
0 & 0 & a^3-3a^2x+3ax^2-x^3(a-x, \\
\underline{a} & \underline{a^2} & \underline{a^3} \qquad [\text{the cube root} \\
a & a^2) & -3a^2x+3ax^2-x^3 \\
\underline{a} & \underline{2a^2} & \underline{-3a^2x+3ax^2-x^3} \\
2a & 3a^2 & \\
\underline{a} & \underline{\quad -3ax+x^2} & \\
3a & 3a^2-3ax+x^2) & \\
\underline{\quad -x} & & \\
\underline{3a-x} & &
\end{array}$$

$$\begin{array}{llll}
(14.) & 0 & 0 & 1-12a+48a^2-64a^3(1-4a, \\
& \underline{1} & \underline{1} & \underline{1} \qquad [\text{the cube root} \\
& 1 & 1) & -12a+48a^2-64a^3 \\
& \underline{1} & \underline{2} & \underline{-12a+48a^2-64a^3} \\
& 2 & 3 & \\
& \underline{1} & \underline{\quad -12a+16a^2} & \\
& 3 & 3-12a+16a^2) & \\
& \underline{\quad -4a} & & \\
& \underline{3-4a} & &
\end{array}$$

The first column of work, in operations of this kind, may obviously be shortened by simply *annexing* the new root-term, instead of writing it underneath, and going through the formality of addition: thus, to the $3a$ in ex. 13 the new term $-x$ might have been annexed; and to the 3 in ex. 14, the $-4a$ might have been annexed. In each of the following examples (16, 17), the first column is shortened in this way.

(15.)

$$\begin{array}{lll}
0 & 0 & 8x^3+36ax^2+54a^2x+27a^3(2x+3a \\
2x & 4x^2 & 8x^3 \\
\hline
2x & 4x^2) & 36ax^2+54a^2x+27a^3 \\
2x & 8x^2 & 36ax^2+54a^2x+27a^3 \\
\hline
4x & 12x^2 & \\
2x & \quad +18ax+9a^2 & \\
\hline
6x & 12x^2+18ax+9a^2) & \\
\quad +3a & & \\
\hline
6x+3a & &
\end{array}$$

(16.)

$$\begin{array}{lll}
0 & 0 & x^6-6x^5+15x^4-20x^3+15x^2-6x+1 \\
x^2 & x^4 & x^6 \qquad (x^2-2x+1 \\
\hline
x^2 & x^4) & -6x^5+15x^4-20x^3 \\
x^2 & 2x^4 & -6x^5+12x^4-8x^3 \\
\hline
x^2 & 3x^4 & 3x^4-12x^3+15x^2-6x+1 \\
2x^2 & -6x^3+4x^2 & 3x^4-12x^3+15x^2-6x+1 \\
\hline
3x^2-2x & 3x^4-6x^3+4x^2) & \\
\quad -2x & \quad -6x^3+8x^2 & \\
\hline
3x^2-4x & 3x^4-12x^3+12x^2 & \\
\quad -2x & \qquad 3x^2-6x+1 & \\
\hline
3x^2-6x+1 & 3x^4-12x^3+15x^2-6x+1) &
\end{array}$$

(17.)

$$\begin{array}{lll}
0 & 0 & y^6-6y^5+6y^4+16y^3-13y^2-24y-8(y^2 \\
y^2 & y^4 & y^6 \qquad [-2y-2 \\
\hline
y^2 & y^4) & -6y^5+6y^4+16y^3 \\
y^2 & 2y^4 & -6y^5+12y^4-8y^3 \\
\hline
2y^2 & 3y^4 & -6y^4+24y^3-12y^2-24y-8 \\
y^2 & -6y^3+4y^2 & -6y^4+24y^3-12y^2-24y-8 \\
\hline
3y^2-2y & 3y^4-6y^3+4y^2) & \\
\quad -2y & \quad -6y^3+8y^2 & \\
\hline
3y^2-4y & 3y^4-12y^3+12y^2 & \\
\quad -2y & \qquad -6y^2+12y+4 & \\
\hline
3y^2-6y-2 & 3y^4-12y^3+6y^2+12y+4) &
\end{array}$$

NOTE.—The above method of extracting the cube root may be abridged; but the uniformity and extreme simplicity of the several steps render them so easy of performance, and of retention in the memory, that any curtailment of the work would be injudicious. (See Young's "Rudimentary Arithmetic," p. 151.)

SURDS (Page 49).

(10.) This expression is the same as $\sqrt{(4 . 3)}+\sqrt{(9 . 3)}-\sqrt{3}+\sqrt{(16 . 3)}=2\sqrt{3}+3\sqrt{3}-\sqrt{3}+4\sqrt{3}=8\sqrt{3}$.

(11.) This is the same as $\sqrt[3]{(8 . 5)}-3\sqrt[3]{(64 . 5)}+4\sqrt[3]{(27 . 5)}=2\sqrt[3]{5}-3 . 4\sqrt[3]{5}+4 . 3\sqrt[3]{5}=2\sqrt[3]{5}$.

$$(12.)\ \sqrt[3]{16}-6\sqrt[3]{\frac{1}{4}}+2\sqrt[3]{54}-4\sqrt[3]{\frac{1}{32}}=$$

$$\sqrt[3]{16}-6\sqrt[3]{\frac{2}{8}}+2\sqrt[3]{54}-4\sqrt[3]{\frac{2}{64}}=$$

$$\sqrt[3]{(8 . 2)}-\frac{6}{2}\sqrt[3]{2}+2\sqrt[3]{(27 . 2)}-\frac{4}{4}\sqrt[3]{2}=$$

$$2\sqrt[3]{2}-3\sqrt[3]{2}+2 . 3\sqrt[3]{2}-\sqrt[3]{2}=4\sqrt[3]{2}.$$

$$(13.)\ 2\sqrt{\frac{a}{2}}-b\sqrt{\frac{a}{b}}=2\sqrt{\frac{2a}{4}}-b\sqrt{\frac{ab}{b^2}}=\frac{2}{2}\sqrt{2a}-$$

$$\frac{b}{b}\sqrt{ab}=\sqrt{2a}-\sqrt{ab}.$$

(14.) $5\sqrt{a^3}-(4a)^{\frac{1}{2}}-3a^{\frac{3}{2}}+\sqrt{(16a)}=5a^{\frac{3}{2}}-2a^{\frac{1}{2}}-3a^{\frac{3}{2}}+4a^{\frac{1}{2}}=$
$2a^{\frac{3}{2}}+2a^{\frac{1}{2}}=2a^{\frac{1}{2}}(a+1)$.

(15.) $2\sqrt[3]{(2x)}+6\sqrt[6]{(4x^2)}+\sqrt[9]{(8x^3)}=2\sqrt[3]{(2x)}+6\sqrt[3]{(2x)}+$
$\sqrt[3]{(2x)}=9\sqrt[3]{(2x)}$.

(16.) $\sqrt{(18a^5b^3)}+\sqrt{(50a^3b^3)}=\sqrt{(9a^4b^2 . 2ab)}+\sqrt{(25a^2b^2 . 2ab)}=3a^2b\sqrt{(2ab)}+5ab\sqrt{(2ab)}=(3a^2b+5ab)\times\sqrt{(2ab)}$.

$$(17.)\ \sqrt[3]{4}\times 7\sqrt[3]{6}\times\sqrt[3]{\frac{5}{8}}=7\sqrt[3]{(4\times 6\times\frac{5}{8})}=7\sqrt[3]{15}.$$

$$(18.)\ \left.\begin{array}{l}(1+\sqrt{5})(1-\sqrt{5})=1-5=-4\\ (\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})=3-2=1\end{array}\right\}\text{(Theorem 3, p. 9, Algebra.)}$$

(19.) Multiply numerator and denominator by $\sqrt{a}+\sqrt{b}$, in order to render the latter rational; then,

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}=\frac{(\sqrt{a}+\sqrt{b})^2}{a-b}=\frac{a+2a^{\frac{1}{2}}b^{\frac{1}{2}}+b}{a-b}.$$

$$\left(\frac{1}{x^3y^2}\right)^{\frac{1}{6}}.x^{\frac{2}{5}}.y^{\frac{1}{3}}=\frac{1}{x^{\frac{3}{6}}y^{\frac{2}{6}}}x^{\frac{2}{5}}y^{\frac{1}{3}}=\frac{x^{\frac{2}{5}}y^{\frac{1}{3}}}{x^{\frac{1}{2}}y^{\frac{1}{3}}}=\frac{x^{\frac{2}{5}}}{x^{\frac{1}{2}}}=\frac{x^{\frac{4}{10}}}{x^{\frac{5}{10}}}=\frac{1}{x^{\frac{1}{10}}}.$$

(20.) Multiply the terms of the fraction by $\sqrt{2}-1$, in order to make the denominator rational: then,

$$\frac{\sqrt{3}-\sqrt{1}}{\sqrt{2}+1}=\frac{(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)}{2-1}=\sqrt{2}-\sqrt{3}+\sqrt{6}-2,$$ as shown by the operation in the margin:

$$\begin{array}{r}
\sqrt{3}-\sqrt{2} \\
\sqrt{2}-\ \ 1 \\
\hline
\sqrt{6}-\ \ 2 \\
-\sqrt{3}+\sqrt{2} \\
\hline
\sqrt{6}-\sqrt{3}+\sqrt{2}-2 \\
\hline
\end{array}$$

$$(\sqrt{3}+\sqrt{2})^2=3+2\sqrt{6}+2=5+2\sqrt{6}$$

$$\begin{array}{r}
\sqrt{3}+\sqrt{2} \\
\hline
5\sqrt{3}+2\sqrt{18} \\
5\sqrt{2}+2\sqrt{12} \\
\hline
\end{array}$$

$$5\sqrt{3}+2\sqrt{(9.2)}+5\sqrt{2}+2\sqrt{(4.3)}=5\sqrt{3}+2.3\sqrt{2}+5\sqrt{2}+2.2\sqrt{3}=9\sqrt{3}+11\sqrt{2}.$$

(21.) These fractions reduced to a common denominator become, when added,

$$\frac{x+\sqrt{(x^2+1)}+x-\sqrt{(x^2+1)}}{x^2-(x^2-1)}=\frac{2x}{1}=2x.$$

Multiply numerator and denominator of the next fraction by the numerator: then,

$$\frac{\sqrt{(x+y)}+\sqrt{(x-y)}}{\sqrt{(x+y)}-\sqrt{(x-y)}}=\frac{\{\sqrt{(x+y)}+\sqrt{(x-y)}\}^2}{(x+y)-(x-y)}=$$

$$\frac{x+y+2\sqrt{(x^2-y^2)}+x-y}{2y}=\frac{2x+2\sqrt{(x^2-y^2)}}{2y}=\frac{x}{y}+\frac{\sqrt{(x^2-y^2)}}{y}$$

(22.) To prove the first, we have only to square the proposed expression; thus:

$$\left\{\frac{\sqrt{(a^2-x^2)}}{2}+\frac{x}{2}\right\}^2=\frac{a^2-x^2}{4}+\frac{x\sqrt{(a^2-x^2)}}{2}+\frac{x^4}{4}$$
$$=\frac{a^2}{4}+\frac{x}{2}\sqrt{(a^2-x^2)}.$$

To prove the second, we have, by writing down the square,

$$\frac{x+\sqrt{(x^2-a)}}{2}+\sqrt{[x^2-(x^2-a)]}+\frac{x-\sqrt{(x^2-a)}}{2}=\frac{2x}{2}+\sqrt{a}$$
$$=x+\sqrt{a}.$$

SIMPLE EQUATIONS (Page 57).

(1.) $\sqrt{(x^2+3)}=\sqrt{7}$. Squaring each side, $x^2+3=7$. $\therefore x^2=4 \therefore x=2$.

NOTE.—It should be observed, that $(-2)^2$ is 4, as well as $(+2)^2$; the square root of 4 is as much -2 as $+2$: the value of x in this example is, therefore, either $+2$ or -2, which is usually expressed thus: $x=\pm 2$. (See p. 64, Algebra.)

(2.) $\sqrt{(x^2-16)}=x-2$. Squaring, $x^2-16=x^2-4x+4$. Transposing, $4x=16+4=20 \therefore x=5$.

(3.) $\sqrt{x}-1=\sqrt{(x-9)}$. Squaring, $x-2\sqrt{x}+1=x-9$. Transposing, $-2\sqrt{x}=-10$. Squaring, $4x=100 \therefore x=25$.

(4.) $\sqrt{x}+\sqrt{(x-3)}=3$. Transposing, $\sqrt{(x-3)}=3-\sqrt{x}$. Squaring, $x-3=9-6\sqrt{x}+x$. Transposing, $6\sqrt{x}=12 \therefore \sqrt{x}=2$. Squaring, $x=4$.

(5.) $\sqrt{x}-\sqrt{2}=\sqrt{(x-2)}$. Squaring, $x-2\sqrt{2x}+2=x-2$. Transposing, $2\sqrt{2x}=4 \therefore \sqrt{2x}=2$. Squaring, $2x=4 \therefore x=2$.

(6.) $\sqrt{(4x+3)}=3$. Cubing, $4x+3=27 \therefore 4x=24 \therefore x=6$.

(7.) $\sqrt{(5x+4)}=\sqrt{3x}+2$. Squaring, $5x+4=3x+4\sqrt{3x}+4$. Transposing, $2x=4\sqrt{3x} \therefore x=2\sqrt{3x}$. Squaring, $x^2=12x \therefore x=12$.

(8.) $\sqrt{(x^2+a^2)}+x=b$. Transposing, $\sqrt{(x^2+a^2)}=b-x$. Squaring, $x^2+a^2=b^2-2bx+x^2$. Transposing, $2bx=b^2-a^2$ $\therefore x=\frac{b^2-a^2}{2b}$.

(9.) $2\sqrt{x}-\sqrt{a}=2\sqrt{(x-a)}$. Squaring, $4x-4\sqrt{ax}+a=4(x-a)$. Transposing, $5a=4\sqrt{ax}$. Squaring, $25a^2=16ax$
$$\therefore \frac{25a}{16}=x.$$

(10.) $a+x=\sqrt{(x^2+5x-a)}$. Squaring, $a^2+2ax+x^2=x^2+5x-a$. Transposing, $a^2+a=(5-2a)x \therefore x=\frac{a^2+a}{5-2a}$.

(11.) $\sqrt{(a^2+x^2)}=\sqrt[4]{(b^4+x^4)}$. Squaring, $a^2+x^2=\sqrt{(b^4+x^4)}$. Squaring again, $a^4+2a^2x^2+x^4=b^4+x^4$. Transposing, $2a^2x^2=b^4-a^4 \therefore x^2=\frac{b^4-a^4}{2a^2} \therefore x=\sqrt{\frac{b^4-a^4}{2a^2}}$.

(12.) $\sqrt{(a-x)}=\frac{a}{\sqrt{(a-x)}}-x$. Multiplying by $\sqrt{(a-x)}$, $a-x=a-x\sqrt{(a-x)}$. Transposing, $x\sqrt{(a-x)}=x \therefore \sqrt{(a-x)}=1$. Squaring, $a-x=1 \therefore a-1=x$.

(13.) $\sqrt{(a+x)}+\sqrt{(a-x)}=2\sqrt{x}$. Squaring, $a+x+2\sqrt{(a^2-x^2)}+a-x=4x$. Transposing, $2\sqrt{(a^2-x^2)}=4x-2a \therefore \sqrt{(a^2-x^2)}=2x-a$. Squaring, $a^2-x^2=4x^2-4ax+a^2$. Transposing, $5x^2-4ax=0$. Dividing by x, $5x-4a=0 \therefore 5x=4a \therefore x=\frac{4}{5}a$.

(14.) $\frac{\sqrt{x}-2}{3}+3=\frac{x-4}{\sqrt{x}+2}$. The numerator of this last fraction is $(\sqrt{x}+2)(\sqrt{x}-2)$: hence the equation is $\frac{\sqrt{x}-2}{3}+3=\sqrt{x}-2$. Subtracting the first fraction from each side, $3=\frac{2(\sqrt{x}-2)}{3} \therefore 9=2\sqrt{x}-4 \therefore 13=2\sqrt{x} \therefore$ Squaring, $4x=169 \therefore x=\frac{169}{4}=42\frac{1}{4}$.

(15.) $x-\sqrt{\{a^2+x\sqrt{(x^2-1)}\}}=a$. Transposing, $\sqrt{\{a^2+x\sqrt{(x^2-1)}\}}=x-a$. Squaring, $a^2+x\sqrt{(x^2-1)}=x^2-2ax+a^2$. Transposing, $x\sqrt{(x^2-1)}=x^2-2ax$. Dividing by x, $\sqrt{(x^2-1)}=x-2a$. Squaring, $x^2-1=x^2-4ax+4a^2$. Transposing, $4ax=4a^2+1 \therefore x=\frac{4a^2+1}{4a}$

(16.) $\dfrac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}-\sqrt{(a-x)}}=\dfrac{c}{1}$. It is proved in the Note below, that if two fractions are equal, the sum of the numerator and denominator divided by their difference is the same for each fraction. Applying this principle in the present case, we have

$$\frac{\sqrt{(a+x)}}{\sqrt{(a-x)}}=\frac{c+1}{c-1}.\text{ Squaring, }\frac{a+x}{a-x}=\frac{(c+1)^2}{(c-1)^2}.$$

Clearing fractions, $(c-1)^2(a+x)=(c+1)^2(a-x)$.

Transposing, $\{(c-1)^2+(c+1)^2\}x=\{(c+1)^2-(c-1)^2\}a$:

that is, $(2c^2+2)x=4ac$

$$\therefore x=\frac{2ac}{c^2+1}.$$

NOTE.—The principle employed in the solution of this equation is often of considerable use in simplifying the work; it may be proved as follows: let the two equal fractions be

$\dfrac{m}{n}=\dfrac{p}{q}$; then, $\dfrac{m}{n}+1=\dfrac{p}{q}+1$: that is, $\dfrac{m+n}{n}=\dfrac{p+q}{q}$....... (A)

Also, $\dfrac{m}{n}-1=\dfrac{p}{q}-1$: ...... $\dfrac{m-n}{n}=\dfrac{p-q}{q}$....... (B

Divide (A) by (B); then $\dfrac{m+n}{m-n}=\dfrac{p+q}{p-q}$: which is the principle employed above.

(17.) $\sqrt{(x+a)}=\sqrt{a}+\sqrt{(x-a)}$. Squaring, $x+a=a+2\sqrt{(ax-a^2)}+x-a$. Transposing, $a=2\sqrt{(ax-a^2)}$. Squaring, $a^2=4(ax-a^2)$. Transposing again, $5a^2=4ax \therefore x=\dfrac{5a}{4}$.

(18.) $\sqrt{x}-\sqrt{(a-x)}=\dfrac{\sqrt{x}+\sqrt{(a-x)}}{2}$. Transposing,

$\dfrac{\sqrt{x}-3\sqrt{(a-x)}}{2}=0 \therefore \sqrt{x}-3\sqrt{(a-x)}=0 \therefore \sqrt{x}=3\sqrt{(a-x)}$

$\therefore x=9(a-x) \therefore 10x=9a \therefore x=\dfrac{9a}{10}$.

(19.) $\sqrt{\left(\dfrac{a^2}{x}+b\right)}-\sqrt{\left(\dfrac{a^2}{x}-b\right)}=c$. Squaring,

$$\frac{a^2}{x}+b-2\sqrt{\left(\frac{a^4}{x^2}-b^2\right)}+\frac{a^2}{x}-b=c^2.$$

$$\text{Transposing, } 2\sqrt{\left(\frac{a^4}{x^2}-b^2\right)}=\frac{2a^2}{x}-c^2.$$

$$\text{Squaring, } 4\left(\frac{a^4}{x^2}-b^2\right)=\frac{4a^4}{x^2}-\frac{4a^2c^2}{x}+c^4.$$

Transposing, $\frac{4a^2c^2}{x}=4b^2+c^4 \therefore 4a^2c^2=(4b^2+c^4)x \therefore x=$

$$\frac{4a^2c^2}{4b^2+c^4}.$$

(20.) This equation was probably intended to be $\sqrt{(x^2+9)}=-2+\frac{x^2-9}{\sqrt{(x^2+9)}-3}$. Adding 3 to each side, $\sqrt{(x^2+9)}+3=1+\frac{x^2-9}{\sqrt{(x^2+9)}-3}$. Clearing fraction, $x^2+9-9=\sqrt{(x^2+9)}-3+x^2-9$, Transposing, $12=\sqrt{(x^2+9)} \therefore 144=x^2+9 \therefore x^2=135 \therefore x=3\sqrt{15}$.

(21.) $\frac{na^2}{\sqrt{(x^2+a^2)}}-x=\sqrt{(a^2+x^2)}$.

Clearing, $na^2-x\sqrt{(x^2+a^2)}=x^2+a^2$,
Transposing, $(n-1)a^2-x^2=x\sqrt{(x^2+a^2)}$.
Squaring, $(n-1)^2a^4-2(n-1)a^2x^2+x^4=x^4+a^2x^2$.
Transposing, $(n-1)^2a^4=(2n-1)a^2x^2$

$$\therefore x^2=\frac{(n-1)^2a^2}{(2n-1)} \therefore x=\frac{(n-1)a}{(2n-1)^{\frac{1}{2}}}.$$

(22.) $ax-\sqrt{(x^2+x+1)}=\sqrt{(x^2-x+1)}$.
Transposing, $ax=\sqrt{(x^2+x+1)}+\sqrt{(x^2-x+1)}$. Squaring, $a^2x^2=x^2+x+1+2\sqrt{\{(x^2+1)+x\}\{(x^2+1)-x\}}+x^2-x+1$: that is, $a^2x^2=2x^2+2+2\sqrt{\{(x^2+1)^2-x^2\}}=2x^2+2+2\sqrt{\{x^4+x^2+1\}}$. Transposing, $(a^2-2)x^2-2=2\sqrt{\{x^4+x^2+1\}}$.

Squaring, $(a^2-2)^2x^4-4(a^2-2)x^2+4=4(x^4+x^2+1)$;
that is, $(a^4-4a^2+4)x^4-4a^2x^2+8x^2=4x^4+4x^2$.
Transposing, $(a^2-4)a^2x^4-4a^2x^2+4x^2=0$.

Dividing by x^2, $(a^2-4)a\,x^2-4a^2+4=0$.

Transposing, $(a^2-4)a^2x^2=4(a^2-1)$

$$\therefore x^2=\frac{4}{a^2}\cdot\frac{a^2-1}{a^2-4}\therefore x=\frac{2}{a}\sqrt{\frac{a^2-1}{a^2-4}}.$$

(23.) $\sqrt{(a^2+ax)}=a-\sqrt{(a^2-ax)}$. Transposing, $\sqrt{(a^2+ax)}+\sqrt{(a^2-ax)}=a$.

Squaring, $a^2+ax+2\sqrt{(a^4-a^2x^2)}+a^2-ax=a^2$.

Transposing, $2a\sqrt{(a^2-x^2)}=-a^2$.

Dividing by a, and squaring, $4a^2-4x^2=a^2$

$$\therefore x^2=\frac{3a^2}{4}\therefore x=\frac{a}{2}\sqrt{3}.$$

(24.) $\frac{1}{x}+\frac{1}{a}=\sqrt{\left\{\frac{1}{a^2}+\sqrt{\left(\frac{4}{b^2x^2}+\frac{1}{x^4}\right)}\right\}}$. Squaring,

$$\frac{1}{x^2}+\frac{2}{ax}+\frac{1}{a^2}=\frac{1}{a^2}+\sqrt{\left(\frac{4}{b^2x^2}+\frac{1}{x^4}\right)}$$

$$\therefore\frac{1}{x^2}+\frac{2}{ax}=\sqrt{\left(\frac{4}{b^2x^2}+\frac{1}{x^4}\right)}.$$

Squaring, $\frac{1}{x^4}+\frac{4}{ax^3}+\frac{4}{a^2x^2}=\frac{4}{b^2x^2}+\frac{1}{x^4}$.

Subtracting $\frac{1}{x^4}$, and then multiplying by $\frac{x^2}{4}$,

$$\frac{1}{ax}+\frac{1}{a^2}=\frac{1}{b^2}.$$

Multiplying by a^2b^2x, $ab^2+b^2x=a^2x$

$$\therefore ab^2=(a^2-b^2)x\therefore x=\frac{ab^2}{a^2-b^2}.$$

(25.) $\frac{ax-1}{\sqrt{ax}+1}-4=\frac{\sqrt{ax}-1}{2}$. In the first fraction, the denominator consists of the sum of two quantities, and the numerator is the difference of the squares of those quantities; hence (Algebra, p. 9), the equation is the same as

$\sqrt{ax}-1-4=\frac{\sqrt{ax}-1}{2}$, or $\sqrt{ax}-5=\frac{\sqrt{ax}-1}{2}\therefore 2\sqrt{ax}-9=\sqrt{ax}\therefore\sqrt{ax}=9\therefore ax=81\therefore x=\frac{81}{a}$.

(26.) $\frac{1}{2}\sqrt{(x^2+3a^2)}+\frac{1}{2}\sqrt{(x^2-3a^2)}=x\sqrt{a}$. Multiply by 2.

$$\sqrt{(x^2+3a^2)}+\sqrt{(x^2-3a^2)}=2x\sqrt{a}.$$

Squaring, $x^2+3a^2+2\sqrt{(x^4-9a^4)}+x^2-3a^2=4ax^2$: that is, $2x^2+2\sqrt{(x^4-9a^4)}=4ax^2$. Divide by 2, and transpose

$$\sqrt{(x^4-9a^4)}=(2a-1)x^2. \text{ Square}$$

$x^4-9a^4=(4a^2-4a+1)x^4$. Transpose and divide by a

$$(4-4a)x^4=9a^3 \therefore x=\sqrt[4]{\frac{9a^3}{4-4a}}.$$

(27.) $\frac{1}{\sqrt{(1-x)}+1}+\frac{1}{\sqrt{(1+x)}-1}=\frac{1}{x}$. Multiply the terms of the first fraction by $1-\sqrt{(1-x)}$, and those of the second by $\sqrt{(1+x)}+1$, and the equation becomes

$$\frac{1-\sqrt{(1-x)}}{x}+\frac{\sqrt{(1+x)}+1}{x}=\frac{1}{x}.$$

Multiplying by x, and transposing,

$$\sqrt{(1-x)}-\sqrt{(1+x)}=1. \text{ Squaring,}$$

$1-x-2\sqrt{(1-x^2)}+1+x=1$: that is, $2-2\sqrt{(1-x^2)}=1$.

Transposing and squaring, $4-4x^2=1 \therefore x^2=\frac{3}{4} \therefore x=\frac{\sqrt{3}}{2}$.

(28.) $\sqrt[3]{(1+x)}+\sqrt[3]{(1-x)}=\sqrt[3]{2}$. Cubing (Algebra, p. 38),

$$(1+x)+(1-x)+3\sqrt[3]{(1-x^2)}\{\sqrt[3]{(1+x)}+\sqrt[3]{(1-x)}\}=2.$$

But from the given condition, the quantity within the braces is equal to $\sqrt[3]{2}$. Consequently,

$$2+3\sqrt[3]{(1-x^2)}\{\sqrt[3]{2}\}=2 \therefore 3\sqrt[3]{(2-2x^2)}=0$$

$$\therefore 2-2x^2=0 \therefore x^2=1 \therefore x=1 \text{ or } -1.$$

(29.) $\frac{x+\sqrt{(x^2-1)}}{x-\sqrt{(x^2-1)}}+\frac{x-\sqrt{(x^2-1)}}{x+\sqrt{(x^2-1)}}=4x(x-1)$.

Multiply the terms of each fraction by the numerator of that fraction, and the equation becomes

$$\{x+\sqrt{(x^2-1)}\}^2+\{x-\sqrt{(x^2-1)}\}^2=4x(x-1):$$

that is, $2x^2+2(x^2-1)=4x^2-4x \therefore 2=4x \therefore x=\frac{1}{2}$.

(30.) $\left.\begin{array}{l} x+y=5 \\ x-y=1 \end{array}\right\}$ By adding and subtracting, we have $2x=6$, $2y=4 \therefore x=3, y=2$.

(31.) $\left.\begin{array}{l}2x-y=1.\\x+3y-11.\end{array}\right\}$ Multiplying the second equation by 2, we have

$$2x-y=1$$
$$2x+6y=22$$

By subtraction, $7y=21 \therefore y=3$

$$\therefore x=11-3y=11-9=2.$$

(32.) $\left.\begin{array}{l}4x-11y=9.\\2x+3y=13.\end{array}\right\}$ Multiplying the second equation by 2,

$$4x-11y=9$$
$$4x+6y=26$$

By subtraction, $17y=17 \therefore y=1$

$$\therefore x=\frac{13-3y}{2}=\frac{10}{2}=5.$$

(33.) $\left.\begin{array}{r}3x+2y=23.\\-2x+5y=29.\end{array}\right\}$ Multiplying the first by 2, and the second by 3,

$$6x+4y=46$$
$$-6x+15y=87$$

By addition, $19y=133 \therefore y=7.$

Again: multiplying the first by 5, and the second by 2,

$$15x+10y=115$$
$$-4x+10y=58$$

By subtraction, $19x \quad = 57 \therefore x=3$

(34.) $\left.\begin{array}{l}\frac{x}{2}-y=1.\\x-\frac{y}{2}=8.\end{array}\right\}$ From the second equation $x=\frac{y}{2}+8$. Substituting this for x in the first, we have $\frac{y}{4}+4-y=1 \therefore 3=\frac{3y}{4} \therefore y=4$

$$\therefore x=\frac{y}{2}+8=2+8=10.$$

(35.) $\left.\begin{array}{l}\frac{x}{3}+\frac{y}{5}=5.\\2x+\frac{y}{3}=17.\end{array}\right\}$ Clearing fractions, the equations become

$$5x+3y=75$$
$$6x+\ y=51 \therefore y=51-6x.$$

Substituting the value of y in the first equation, we have

$$5x+153-18x=75. \quad \text{Transposing, } 78=13x \therefore x=6$$
$$\therefore y=51-6x=51-36=15.$$

(37.) $\left.\begin{array}{l}\dfrac{x+y}{10}+\dfrac{x-y}{2}=0. \\ \dfrac{x+y}{5}+\dfrac{x-y}{2}=1.\end{array}\right\}$ Subtract the first equation from the second, and there results the equation

$$\frac{x+y}{10}=1 \therefore \tfrac{1}{2}(x+y)=5$$

Hence from the first equation,

$$1+\tfrac{1}{2}(x-y)=0 \therefore \tfrac{1}{2}(x-y)=-1$$

By adding and subtracting, $\left\{\begin{array}{lcr} x & = & 4 \\ y & = & 6\end{array}\right.$

(38.) $\left.\begin{array}{l}\dfrac{2x-y}{4}-\dfrac{3}{2}=\dfrac{3y}{4}-x-2. \\ \dfrac{x+y}{3}=2\tfrac{2}{3}.\end{array}\right\}$ By transposition, the first of these equations becomes

$$\frac{2x-4y}{4}+x=-\tfrac{1}{2};$$

$$\text{that is, } \frac{3x-2y}{2}=-\tfrac{1}{2} \therefore 3x-2y=-1;$$

and, by clearing fractions, the second is $x+y=8$.

From this last equation, $x=8-y$; and this, substituted in the preceding, gives $24-3y-2y=-1 \therefore 5y=25 \therefore y=5$.

(39.) $\left.\begin{array}{l} ax+by=c. \\ \dfrac{x}{b}-\dfrac{y}{a}=1.\end{array}\right\}$ Multiplying the second by ab, the equations are

$$\begin{array}{l} ax+by=c \\ ax-by=ab \end{array}$$

$$\text{Adding, } 2ax \quad =ab+c \therefore x=\frac{ab+c}{2a}$$

$$\text{Subtracting, } 2by=c-ab \therefore y=\frac{c-ab}{2b}.$$

(40.) $\left.\begin{array}{l}\dfrac{x+2}{y}=\dfrac{7}{8}. \\ \dfrac{x}{y-2}=\dfrac{5}{6}.\end{array}\right\}$ Clearing fractions, $8x+16=7y$

$$6x=5y-10$$

$$\text{Subtracting, } 2x+16=2y+10$$
$$\therefore 2y-2x=6 \therefore y=x+3.$$

Hence by substitution, $8x+16=7x+21 \therefore x=5 \therefore y=8$.

(41.) $\left.\begin{array}{l}\dfrac{m}{x}+\dfrac{n}{y}=a. \\ \dfrac{n}{x}+\dfrac{m}{y}=b.\end{array}\right\}$ Multiply the first by m, the second by n, and subtract: we thus have

$$\frac{m^2-n^2}{x}=ma-nb \therefore x=\frac{m^2-n^2}{ma-nb}.$$

Again: multiply the first by n, the second by m, and subtract; then,

$$\frac{m^2-n^2}{y}=mb-na \therefore y=\frac{m^2-n^2}{mb-na}.$$

(42.) $\left.\begin{array}{l}(x+1)(y-9)=(y+7)(x+5)-112. \\ 3y-2x=9.\end{array}\right\}$ The first equation is

$$xy+y-9x-9=xy+7x+5y+35-112;$$

that is, by transposition, $4y+16x=68$,

so that the equations are

$\left.\begin{array}{l}y+4x=17. \\ 3y-2x=\ \ 9.\end{array}\right\}$ Multiplying the second by 2, and adding, we have

$$7y=18+17=35 \therefore y=5 \therefore 5+4x=17 \therefore x=\frac{17-5}{4}=3.$$

(43.) $\left.\begin{array}{l}y+\dfrac{x}{4}=10-\dfrac{y-2x-1}{3}. \\ \dfrac{2x-1}{10}-\dfrac{6x-2y}{5}=\dfrac{x-y}{10}.\end{array}\right\}$ Clearing fractions, $12y+3x=120-4y+8x+4$

$2x-1-12x+4y=x-y.$

Transposing, $16y-\ 5x=124$

$5y-11x=\ \ 1.$

Multiplying the first of these by 11, and the second by 5,

$$\begin{array}{r}176y-55x=1364 \\ 25y-55x=\ \ \ \ \ 5 \\ \hline\end{array}$$

$$\text{Subtracting, } 151y \qquad =1359 \therefore y=\frac{1359}{151}=9$$

$$\therefore x=\frac{5y-1}{11}=\frac{44}{11}=4.$$

(44.) $\left.\begin{array}{l}3\cdot4x-\cdot02y=\cdot01. \\ 2x+\ \cdot4y=1\cdot2.\end{array}\right\}$ Multiplying the first by 20, we have

$$\begin{array}{rr} & 68x-\cdot4y=\cdot2 \\ \text{Adding,} & 2x+\cdot4y=1\cdot2 \\ \hline\end{array}$$

$$70x \qquad =1\cdot4 \therefore x=\frac{1\cdot4}{70}=\cdot02.$$

and by the second equation, $\cdot 04+\cdot 4y=1\cdot 2$ $\therefore y=\frac{1\cdot 16}{\cdot 4}=2\cdot 9$

(45.) $\left.\begin{array}{l}\frac{x+ay}{3}=c. \\ ax-by=c.\end{array}\right\}$ or $ax+a^2y=3ac$, by multiplying by $3a$,
$ax-by=c$

Subtracting, $(a^2+b)y=(3a-1)c$ $\therefore y=\frac{(3a-1)c}{a^2+b}$

Also, $bx+aby=3bc$ by multiplying the first
$a^2x-aby=ac$ [by $3b$.

Adding, $(a^2+b)x=(a+3b)c$ $\therefore x=\frac{(a+3b)c}{a^2+b}$.

(46.) $\left.\begin{array}{l}ax+by=c^2. \\ a(a+x)=b(b+y).\end{array}\right\}$ Add a^2+b^2 to the first equation, then it becomes

$$a(a+x)+b(b+y)=a^2+b^2+c^2.$$

Also, $a(a+x)-b(b+y)=0$.

Adding and subtracting, $\left\{\begin{array}{l}2a(a+x)=a^2+b^2+c^2 \\ 2b(b+y)=a^2+b^2+c^2\end{array}\right.$

$$\therefore x=\frac{b^2+c^2-a^2}{2a},\ y=\frac{a^2+c^2-b^2}{2b}.$$

(47.) $\left.\begin{array}{l}2x-2y+3z=16. \\ 3x+5y-2z=6. \\ 4x+3y-4z=-1.\end{array}\right\}$ Multiplying the first by 2 and the second by 3, in order to eliminate z, we have

$$4x-4y+6z=32$$
$$9x+15y-6z=18$$

Adding, $13x+11y \quad =50\ldots.$ (A).

Again: multiplying the second equation by 2, we have

$$6x+10y-4z=12$$
$$4x+3y-4z=-1$$

Subtracting, $2x+7y \quad =13\ldots.$ (B).

Multiplying (A) by 7 and (B) by 11, in order to eliminate y, we have

$$91x+77y=350$$
$$22x+77y=143$$

Subtracting, $69x \quad =207 \therefore x=3$

$\therefore$ from (B), $y=\frac{13-2x}{7}=\frac{7}{7}=1$. And substituting these values of x and y in either of the proposed equations, the third unknown z becomes determined thus: taking the second of the given equations, we have

$$z=\frac{3x+5y-6}{2}=\frac{9+5-6}{2}=4.$$

(48.) $\left.\begin{array}{l}\frac{x}{3}+\frac{y}{4}+\frac{z}{5}=47. \\ \frac{x}{4}+\frac{y}{5}+\frac{z}{6}=38. \\ \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=62.\end{array}\right\}$ These equations, cleared of fractions, become the following — namely,

$$20x+15y+12z=2820$$
$$30x+24y+20z=4560$$
$$12x+\ 8y+\ 6z=1488$$

Multiply the first by 3 and the second by 2, then

$$60x+45y+36z=8460$$
$$60x+48y+40z=9120$$

By subtraction, $\quad 3y+\ 4z=\ 660\ldots\ldots\ldots(A)$

Multiply the first by 3 and the third by 5, then

$$60x+45y+36z=8460$$
$$60x+40y+30z=7440$$

By subtraction, $\quad 5y+\ 6z=1020\ldots\ldots\ldots(B)$

Multiply (A) by 3 and (B) by 2, then

$$9y+12z=1980$$
$$10y+12z=2040$$

By subtraction, $\quad y \quad = \quad 60$

$\therefore$ (A) $z=\frac{660-3y}{4}=165-45=120$. And, from the first

of the given equations,

$$x=\frac{2820-15y-12z}{20}=\frac{2820-900-1440}{20}=24.$$

$$(49.)\quad \left.\begin{aligned} \frac{x+y}{z}&=5.\\ \frac{y-z}{x}&=1.\\ \frac{x-z}{y}&=\frac{1}{3}\end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} x+y-5z&=0.\\ y-z-x&=0.\\ 3x-3z-y&=0.\end{aligned}\right.$$

Subtracting the second from the first, we have

$$2x-4z=0 \therefore x=2z$$

Substituting this in the third, $y=3z$.

If these values be substituted in either of the equations, the condition of that equation will be satisfied without limiting z to *any* particular value: hence, z may be *anything*, the accompanying values of x and y being $2z$ and $3z$. Thus, if z be taken equal to 2, then $x=4$, $y=6$, $z=2$; if z be taken $=1$, then $x=2$, $y=3$, $z=1$, &c. Either set of values will satisfy the proposed equations, as is obvious.

$$(50.)\quad \left.\begin{aligned} xyz&=40.\\ xyw&=80.\\ yzw&=200.\\ xzw&=100.\end{aligned}\right\}$$

These four equations may be reduced to three, thus: divide the second by the first $\therefore \frac{xyw}{xyz}=2 \therefore w=2z$: hence by substitution, $xyz=40$, $yz^2=100$, $xz^2=50$

$$\therefore x=\frac{40}{yz} \qquad \therefore \frac{40z}{y}=50 \therefore y=\frac{4}{5}z$$

$$\therefore \frac{4}{5}z^3=100 \therefore z^3=25\times 5=5^3$$

$$\therefore z=5 \therefore y=\frac{4}{5}z=4 \therefore x=\frac{40}{yz}=2,\ w=2z=10.$$

Problems (Page 62).

(1.) Let x represent the number; then by the question,

$$\sqrt{(2x+9)}=5. \quad \text{Squaring, } 2x+9=25$$

$$\therefore 2x=16 \therefore x=8.$$

(2.) Let the number be represented by x; then by the question,

$\sqrt{(5x+4)}=2+\sqrt{3x}$. Squaring, $5x+4=4+4\sqrt{3x}+3x$.
Transposing, $2x=4\sqrt{3x}$ $\therefore$ $x=2\sqrt{3x}$ $\therefore$ $x^2=12x$ $\therefore$ $x=12$.

(3.) Let x represent the number; then by the question,
$\sqrt{(x^2-7)}=7-x$. Squaring, $x^2-7=(7-x)^2=49-14x+x^2$
$\therefore$ $14x=56$ $\therefore$ $x=4$.

(4.) Let x represent the greater number, then $56-x$ is the less; and by the question,

$$x-(56-x)=24 \therefore 2x=56+24=80 \therefore x=40$$
$$\therefore 56-x=16.$$

Hence the two numbers are 40 and 16.

Otherwise.—Let x be one number, and y the other:

then, $x+y=56$
and $x-y=24$.

Add and subtract, $\left.\begin{array}{l}2x=80 \therefore x=40\\ 2y=32 \therefore y=16.\end{array}\right\}$ the numbers sought.

(5.) Let x be one number, then $16-x$ is the other; and by the question,

$$\frac{1}{x}+\frac{1}{16-x}=\frac{2}{x}-\frac{2}{16-x}.$$

Transposing, $\frac{3}{16-x}=\frac{1}{x}$ $\therefore$ $3x=16-x$ $\therefore$ $4x=16$

$$\therefore x=4, \therefore 16-x=12.$$

Hence the numbers are 4 and 12.

Otherwise.—Let the numbers be x and y; then by the question, $x+y=16$

$$\text{and } \frac{1}{x}+\frac{1}{y}=\frac{2}{x}-\frac{2}{y}$$
$$\therefore \frac{3}{y}-\frac{1}{x}=0 \therefore 3x-y=0.$$
$$\text{But } x+y=16$$
$$\text{Adding, } 4x=16 \therefore x=4$$
$$\therefore y=16-x=12.$$

(6.) Suppose there were x ships at first: then there were taken, sunk, and burnt, $\frac{x}{3}+\frac{x}{6}+2$; that is, $\frac{x}{2}+2$, so that there remained $x-\left(\frac{x}{2}+2\right)$; that is, $\frac{x}{2}-2$. Of these, one-seventh were lost in the storm, so that six-sevenths of them—that is, $\frac{6}{7}\left(\frac{x}{2}-2\right)$ were left; and by the question,

$$\frac{6}{7}\left(\frac{x}{2}-2\right)=24 \therefore 6\left(\frac{x}{2}-2\right. =168$$

$$\therefore 3x-12=168 \therefore x-4=56 \therefore x=60.$$

Hence there were 60 ships at first.

(7.) Suppose the ages were x and y years: then by the question,

$$\left.\begin{array}{l} x:y::3:4 \\ \text{and } x-10:y-10::2:3 \end{array}\right\} \text{Multiplying extremes and means (Arithmetic, p. 101).}$$

$$4x=3y \qquad \therefore 4x-3y=0$$

$$3x-30=2y-20 \therefore 3x-2y=10.$$

Subtracting, $x-y=-10 \therefore x=y-10$.

Substituting this in the second, $3y-30-2y=10 \therefore y=40$

$\therefore x=30$.

Hence the ages are 30 and 40 years respectively.

(8.) Suppose x gallons must be added: the worth of these will be $14x$ shillings; so that the worth of the whole $20+36+x$ gallons will be $(180+396+14x)$ shillings. But by the question, the same is worth $12(20+36+x)$ shillings.

$\therefore 576+14x=672+12x$

$\therefore 2x=96 \therefore x=48$, the number of gallons.

(9.) Suppose the price of a sheep was x shillings, and the price of a lamb y shillings; then by the question, reducing to shillings, $12x+20y=580 \therefore 3x+5y=145$

Also, $10x+30y=670 \therefore x+3y=67$

Multiplying this last by 3, $3x+9y=201$

Subtracting the first from it, $4y=56 \therefore y=14$

$\therefore x=67-3y=67-42=25$

Hence the price of a sheep was 25*s.*, and that of a lamb 14*s.*

(10.) Suppose x to be the number of pounds A contributed, and y the number B contributed; then, since the whole stock is to the whole gain as the contribution of each to his proper share of that gain, we have

$$833 : 153 :: x : \frac{153x}{833} \text{ A's share,}$$

$$833 : 153 :: y : \frac{153y}{833} \text{ B's share.}$$

Now by the question,

$$\frac{153x}{833} + 45 = \frac{153y}{833}.$$

Also, $x+y=833 \ldots.$ (1).

Substituting $x+y$ for 833 in the denominators above, and clearing fractions, $153x+45x+45y=153y$.

$\therefore 198x-108y=0 \therefore 11x-6y=0 \ldots.$ (2).

Multiplying (1) by 6, we have $6x+6y=4998$

And adding, $17x=4998 \therefore x=294$.

Also from (1) $y=833-x=833-294=539$.

Hence A contributed £294, and B £539.

(11.) Let x represent the number of sheep, then $\frac{x}{10}$ must be the number of acres ploughed, and $\frac{x}{4}$ the number of acres for pasture; therefore, by the question,

$\frac{x}{10}+\frac{x}{4}=700. \therefore$ Multiplying by 20, $2x+5x=1400 \therefore x=2000$.

(12. Let $\frac{x}{y}$ be the fraction: then by the question,

$x+y=5x \therefore y=4x$: hence the fraction is $\frac{x}{4x}=\frac{1}{4}$.

(13.) Let x represent the number; then by the question,

$$x+1 : x+5 :: x+5 : x+13.$$

Multiplying extremes and means, there results the equation $(x+1)(x+13)=(x+5)^2$; that is, $x^2+14x+13=x^2+10x+25$.

Transposing, $14x-10x=25-13 \therefore 4x=12 \therefore x=3$.

(14.) Suppose A takes x hours, and B, y hours; then, when they meet, A will have travelled $x-a$ hours, and B, $y-b$.

Moreover, A travels at the rate of $\frac{1}{x}$ of the distance in an hour, and B at the rate of $\frac{1}{y}$ the distance in an hour; so that in $x-a$ hours A will have gone $\frac{x-a}{x}$ of the distance, and B, $\frac{y-b}{y}$ of the distance. But these parts of the distance make up the *whole*.

$$\therefore \frac{x-a}{x}+\frac{y-b}{y}=1 \dots (1).$$

Also, since, at the time of meeting, each must have travelled the same number of hours, it follows that the difference between the number of hours occupied by A, and the number occupied by B in going the whole distance, must be $a-b$: hence, for a second equation, we have

$$x-y=a-b \dots (2).$$

From the first equation, by clearing fractions,

$$xy-ay+xy-bx=xy$$

$$\text{that is, } (x-a)y=bx \therefore y=\frac{bx}{x-a}.$$

But from the second equation, $y=x-a+b$

$$\therefore \frac{bx}{x-a}=x-a+b \therefore bx=(x-a)^2+bx-ab$$

$$\therefore (x-a)^2=ab \therefore x-a=a^{\frac{1}{2}}b^{\frac{1}{2}} \therefore x=a+a^{\frac{1}{2}}b^{\frac{1}{2}}.$$

Consequently, $y=x-a+b=a^{\frac{1}{2}}b^{\frac{1}{2}}+b$; and these values may be expressed in the following form—namely,

Number of hours taken by A, $a^{\frac{1}{2}}(a^{\frac{1}{2}}+b^{\frac{1}{2}})$
" " " B, $b^{\frac{1}{2}}(a^{\frac{1}{2}}+b^{\frac{1}{2}})$

NOTE.—The equation (1) would be expressed a little more conveniently by writing it thus:

$$1-\frac{a}{x}+1-\frac{b}{y}=1 \therefore 1-\frac{a}{x}=\frac{b}{y} \therefore (x-a)y=bx.$$

(15.) Suppose he takes $\frac{1}{x}$th of a quart of the first: then he must take $1-\frac{1}{x}$ of a quart of the second. The value of the

first of these portions is $\frac{20}{x}$ pence, and that of the second is $12\left(1-\frac{1}{x}\right)$ pence: but the value of both portions is 14 pence, by the question,

$$\therefore \frac{20}{x}+12-\frac{12}{x}=14 \therefore \frac{8}{x}=2 \therefore 8=2x \therefore x=4.$$

Therefore, he must take $\frac{1}{4}$ of a quart at 20 pence, and $\frac{3}{4}$ of a quart at 12 pence; the former portion being worth 5 pence, and the latter being worth 9 pence.

(16.) Let x represent the first digit, and y the second; then, since the local value of x is ten times the figure x, the number itself will be expressed by $10x+y$: hence the conditions of the question are

$$\left.\begin{aligned} 10x+y&=5x+5y \\ \text{and } 11x+2y&=10y+x \end{aligned}\right\} \therefore \left\{\begin{aligned} &5x-4y=0 \\ &10x-8y=0, \text{ or } 5x-4y=0. \end{aligned}\right.$$

It appears from these two equations, that the two conditions of the question are not independent—that is, that the second condition is substantially only a repetition of the first. The inference therefore is, that *any* two digits, x and y, which satisfy the single condition $5x-4y=0$, or $x=\frac{4}{5}y$ will answer the question: all that is necessary being, that the first digit, x, be four-fifths of the second, y. But the only *digit*, of which four-fifths is also a digit, is evidently 5: this, therefore, must be the value of y; and, consequently, 4 must be the value of x: hence the number is 45.

NOTE.—The learner will have frequent occasion to notice, in solving problems by algebra, that the science will often furnish results much more general and comprehensive than the restrictions of the question admit of being received as answers to *it*. The present example is an instance of this; the algebra justifies our assuming *any* number for y, and then taking four-fifths of that number for x: thus, we may take $y=10$, then $x=8$; or we may take $y=15$, then $x=12$, and so on, to any extent. These values all fulfil the original *algebraical* conditions above—namely, $10x+y=5x+5y$, and $11x+2y=10y+x$; but there is a *restriction* in the question from which these algebraical conditions are wholly free—namely, the only values admissible must each be *a single digit;* so that the values of $y=10$, $y=15$, &c., each consisting of

more than one digit, are inadmissible as answers to the *question*. The learner should always examine whether the algebraical solutions, when there are more than one, are not, some of them, in this way rejective in consequence of limitations in the question not being impressed on the algebraical translation of it.

(17.) Suppose he began with x shillings: then at the end of the first sitting he had, according to the question, $3x-16$ shillings. As, at the second sitting, he lost four-fifths of this, he had only one-fifth of it left, which sum, added to the x shillings he afterwards won, amounted by the question to 80 shillings: hence this equation

$$\tfrac{1}{5}(3x-16)+x=80 \therefore 3x-16+5x=400$$
$$\therefore 8x=416 \therefore x=52.$$

Consequently he began with 52 shillings.

(18.) Suppose they can finish it in x days: then they can do $\frac{1}{x}$ of it in 1 day; but by the question A alone can do $\frac{1}{a}$ of it in 1 day, and B alone $\frac{1}{b}$ of it; so that *together* they can do $\frac{1}{a}+\frac{1}{b}$: hence the equation,

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{x} \therefore ax+bx=ab \therefore x=\frac{ab}{a+b}.$$

(19.) Suppose his original stock was x pounds: then he had, at the end of the first year,

$$x-50+\tfrac{1}{3}(x-50), \text{ or } \tfrac{4}{3}(x-50);$$

at the end of the second year,

$$\tfrac{4}{3}(x-50)-50+\tfrac{4}{9}(x-50)-\frac{50}{3}=\frac{16}{9}(x-50)-\frac{200}{3};$$

at the end of the third year,

$$\tfrac{4}{3}\left\{\frac{16}{9}(x-50)-\frac{200}{3}-50\right\}.$$

Hence by the question,

$$\tfrac{4}{3}\left\{\frac{16}{9}(x-50)-\frac{200}{3}-50\right\}=2x$$
$$\therefore \frac{64(x-50)}{9}-\frac{800}{3}-200=6x$$
$$\therefore 64x-3200-2400-1800=54x.$$
$$\therefore 10x=7400 \therefore x=740, \text{ the amount of stock.}$$

NOTE.—A little consideration of this question will show that the stock at the end of the third year must be

$$\tfrac{4}{3}\,.\,\tfrac{4}{3}\,.\,\tfrac{4}{3}(x-50)-\tfrac{4}{3}\,.\,\tfrac{4}{3}\,.\,50-\tfrac{4}{3}\,.\,50.$$

And generally, if the term be n years instead of three, the stock at the end of that term would be found by prefixing $\frac{4}{3}$, as factor, n times to $(x-50)$, $n-1$ times to 50, then $n-2$ times to 50, and so on, till we came to simply $\frac{4}{3}\,.\,50$; each of the products, after the first, to be subtracted.

(20.) Suppose the velocity of the strongest tide to be x miles an hour, and the velocity the man could give the boat without any tide at all y miles an hour; then with the stronger tide in his favour he goes $x+y$ miles an hour, and therefore $\frac{3}{4}\ (x+y)$ in $\frac{3}{4}$ of an hour; and with the weaker tide against him, he goes $y-\frac{1}{2}x$ miles an hour, and consequently $\frac{3}{2}(y-\frac{1}{2}x)$ in $1\frac{1}{2}$ hour: hence, as the distance is the same there and back,

$$\therefore\ \tfrac{3}{4}(x+y)=\tfrac{3}{2}(y-\tfrac{1}{2}x)$$

$$\therefore\ 3x+3y=6y-3x\ \therefore\ 6x=3y.$$

But, by the question,

$$\tfrac{3}{4}(x+y)=5\ \therefore\ 3x+3y=20\ \therefore\ 3y=20-3x$$

$$\therefore\ 6x=20-3x\ \therefore\ 9x=20\ \therefore\ x=2\tfrac{2}{9}.$$

(21.) Let the three parts be x, y, z; then by the question we have these three equations; namely,

$$\left.\begin{array}{l} x+y+z=11520 \\ 9x+9y=7y+7z \\ 8y-8z=x+z \end{array}\right\}\ \therefore\ \left\{\begin{array}{l} x+y+2=11520 \\ 9x+2y-7z=0 \\ -9x+8y-z=0. \end{array}\right.$$

Adding the last two equations, $10y-8z=0$.

Subtracting the second from 9 times the first, $7y+16z=103680$.

Adding twice the first of these to the second, $27y=103680$

$$\therefore\ y=3840\ \therefore\ z=\frac{10y}{8}=\frac{5y}{4}=960\times 5=4800,$$

and $x=11520-(y+z)=2880$.

Hence the three parts are 2880, 3840, and 4800.

(22.) Suppose he bought x sheep: then by the question the cost of each was $\dfrac{94}{x}$ pounds. After his loss he had $x-7$,

and the cost price of one-fourth of these was $\frac{x-7}{4} \cdot \frac{94}{x}$ or $\frac{x-7}{2} \cdot \frac{47}{x}$; but by the question this price is £20: therefore,

$$\frac{47(x-7)}{2x}=20 \therefore 47x-329=40x$$

$$\therefore 7x=329 \therefore \quad x=47, \text{ the number of sheep.}$$

(23.) Suppose x quarters of wheat and y quarters of barley pay the rent: then by the question, $55x=33y$, also $65x+41y=55x+33y+140$, seeing that there are 140 shillings in £7: hence the equations,

$$\left.\begin{matrix} 55x-33y=0 \\ 10x+\ 8y=140 \end{matrix}\right\} \therefore \left\{\begin{matrix} 5x-3y=0 \\ 5x+4y=70. \end{matrix}\right.$$

Subtracting, $7y=70 \therefore y=10$

$$\therefore x=\frac{3y}{5}=6.$$

Hence there were 6 quarters of wheat, and 10 quarters of barley.

(24.) Suppose the distance travelled to be x feet: then the fore-wheel must have revolved $\frac{x}{a}$ times, and the hind-whel $\frac{x}{b}$ times; and by the question,

$$\frac{x}{a}=\frac{x}{b}+n \therefore bx=ax+abn \therefore (b-a)x=abn \therefore x=\frac{abn}{b-a}.$$

PURE QUADRATIC EQUATIONS (Page 65).

(1.) $x^2=144$. Extracting the square root, $x=\pm 12$.

(2.) $x^2-9=16$. Transposing, $x^2=25$. Extracting the square root, $x=\pm 5$.

(3.) $\frac{3x^2}{4}-5=7$. Clearing, $3x^2-20=28$. Transposing, and dividing by 3, $x^2=16$. Extracting the square root, $x=\pm 4$.

(4.) $2\sqrt{(1-x)^2}=\sqrt{3}$. Squaring, $4(1-x^2)=3$, or $4-4x^2=3$.

Transposing, $4x^2=1$. Extracting the square root,
$2x=\pm 1 \therefore x=\pm\frac{1}{2}$.

(5.) $\frac{\sqrt{(a^2+x^2)}-x}{x}=\frac{1}{b} \therefore \frac{\sqrt{(a^2+x^2)}}{x}=\frac{1}{b}+1=\frac{b+1}{b}$.

Squaring, $\frac{a^2+x^2}{x^2}=\frac{(b+1)^2}{b^2} \therefore \frac{a^2}{x^2}=\frac{(b+1)^2}{b^2}-1=\frac{2b+1}{b^2}$.

Or, reversing the terms of each fraction,

$$\frac{x^2}{a^2}=\frac{b^2}{2b+1} \therefore x^2=\frac{a^2b^2}{2b+1}.$$

Hence, extracting the square root, $x=\frac{ab}{\sqrt{(2b+1)}}$.

(6.) $\frac{7x^2}{4}-\frac{4x^2+5}{2}+\frac{2x^2-15}{4}=0$. Multiplying by 4,
$7x^2-8x^2-10+2x^2-15=0 \therefore x^2=25 \therefore x=\pm 5$.

(7.) $\frac{1}{2x^2}+7=\frac{9}{4x^2}$. Multiplying by $4x^2$,
$2+28x^2=9 \therefore 28x^2=7 \therefore 4x^2=1 \therefore 2x=\pm 1 \therefore x=\pm\frac{1}{2}$.

(8.) $35-\frac{x^2+50}{5}=x^2-\frac{x^2-10}{3}$. Multiplying by 15,
$525-3x^2-150=15x^2-5x^2+50$.
Transposing, $325=13x^2 \therefore x^2=25 \therefore x=\pm 5$.

Adfected Quadratics (Page 73).

(1.) $x^2-8x=9$. Completing the square, $x^2-8x+4^2=25$.
Extracting the root, $x-4=\pm 5 \therefore x=4\pm 5=9$ or -1.

(2.) $x^2+12x-16=92 \therefore x^2+12x=108$.
Completing the square, $x^2+12x+6^2=108+36=144$.
Extracting the root, $x+6=\pm 12 \therefore x=-6\pm 12=6$ or -18.

(3.) $x^2-3x=10$. Completing the square,

$$x^2-3x+\left(\frac{3}{2}\right)^2=10+\frac{9}{4}=\frac{49}{4}.$$

Extracting the root, $x-\frac{3}{2}=\pm\frac{7}{2} \therefore x=\frac{3\pm 7}{2}=5$ or -2.

(4.) $x^2-x+3=45$. Transposing, $x^2-x=42$.

Completing the square, $x^2-x+\frac{1}{4}=42\frac{1}{4}=\frac{169}{4}$.

Extracting the root, $x-\frac{1}{2}=\pm\frac{13}{2}\ \therefore\ =\frac{1\pm13}{2}=7$ or -6.

(5.) $5x^2+x=4$. Dividing by 5, $x^2+\frac{x}{5}=\frac{4}{5}$.

Completing the square, $x^2+\frac{x}{5}+\frac{1}{100}=\frac{4}{5}+\frac{1}{100}=\frac{81}{100}$.

Extracting the root, $x+\frac{1}{10}=\pm\frac{9}{10}\ \therefore\ x=\frac{-1\pm9}{10}=\frac{4}{5}$ or -1.

(6.) $2x^2-x=21$. Dividing by 2, $x^2-\frac{x}{2}=\frac{21}{2}$.

Completing the square, $x^2-\frac{x}{2}+\frac{1}{16}=\frac{21}{2}+\frac{1}{16}=\frac{169}{16}$.

Extracting the root, $x-\frac{1}{4}=\pm\frac{13}{4}\ \therefore\ x=\frac{1\pm13}{4}=\frac{7}{2}$ or -3.

(7.) $5x^2+6x-3=60\ \therefore\ 5x^2+6x=63$. Dividing by 5,

$x^2+\frac{6x}{5}=\frac{63}{5}$. Completing the square,

$$x^2+\frac{6x}{5}+\frac{9}{25}=\frac{63}{5}+\frac{9}{25}=\frac{324}{25}.$$

Extracting the root, $x+\frac{3}{5}=\pm\frac{18}{5}\ \therefore\ x=\frac{-3\pm18}{5}=3$ or $-\frac{21}{5}$.

(8.) $x-1=-\frac{1}{x}$. Multiplying by x, $x^2-x=-1$.

Completing the square, $x^2-x+\frac{1}{4}=-\frac{3}{4}$.

Extracting the root, $x-\frac{1}{2}=\frac{\sqrt{-3}}{2}\ \therefore\ x=\frac{1\pm\sqrt{-3}}{2}$.

(9.) $(x-12)(x+2)=0$; that is, $x^2-10x=24$.

Completing the square, $x^2-10x+25=49$.

Extracting the root, $x-5=\pm7\ \therefore\ x=5\pm7=12$ or -2.

Note.—The given equation in this example may be readily solved without going through the process for a quadratic: for

it is plain that the equation $(x-12)(x+2)=0$ is satisfied by equating either of the two factors in the first member of it to 0; that is, it is satisfied for $x-12=0$, and also for $x+2=0$; therefore, it is satisfied for $x=12$, and $x=-2$; and these are the only values that will answer, since a product cannot become nothing except one of its factors become nothing. The learner will thus perceive, that when the factors of the first member of an equation are actually exhibited, the second member being zero, all the roots of that equation are discoverable by simply equating each factor to 0. Thus, if

$$(x-4)(x+6)(x+2)(x-8)=0,$$

the roots or values of x in this equation are $x=4$, $x=-6$, $x=-2$, $x=8$.

Moreover, when one of the roots is known, if the first member be divided by x *minus* that root, the quotient, equated to 0, will be the equation containing the remaining roots. Or if the final or absolute term of an equation, whose terms are all brought to one side, and the coefficient of the highest power of x rendered equal to unity, be divided by one root, the quotient will be the product of all the other roots. The learner will find these principles of frequent application in the following solutions.

(10.) $3x^2-14x=-15$. Dividing by 3, $x^2-\frac{14x}{3}=-5$.

Completing the square, $x^2-\frac{14x}{3}+\frac{49}{9}=\frac{49}{9}-5=\frac{4}{9}$.

Extracting the root, $x-\frac{7}{3}=\pm\frac{2}{3}$ $\therefore x=\frac{7\pm2}{3}=3$ or $1\frac{2}{3}$.

(11.) $2x^2-11x=21$. Dividing by 2, $x^2-\frac{11x}{2}=\frac{21}{2}$.

Completing the square, $x^2-\frac{11x}{2}+\frac{121}{16}=\frac{21}{2}+\frac{121}{16}=\frac{289}{16}$.

Extracting the root, $x-\frac{11}{4}=\pm\frac{17}{4}$ $\therefore x=\frac{11\pm17}{4}=7$ or $-1\frac{1}{2}$.

(12.) $ax^2-bx=c$. Dividing by a, $x^2-\frac{b}{a}x=\frac{c}{a}$.

Completing the square, $x^2-\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{c}{a}+\frac{b^2}{4a^2}=\frac{b^2+4ac}{4a^2}$.

Extracting the root, $x-\frac{b}{2a}=\frac{\sqrt{(b^2+4ac)}}{2a}$

$\therefore x=\frac{b\pm\sqrt{(b^2+4ac)}}{2a}$.

(13.) $4x-\frac{14-x}{x+1}=14$. Clearing, $4x^2+4x-14+x=14x+14$.

Transposing, $4x^2-9x=28$. Dividing by 4, $x^2-\frac{9x}{4}=7$.

Completing the square, $x^2-\frac{9x}{4}+\frac{81}{64}=7+\frac{81}{64}=\frac{529}{64}$.

Extracting the root, $x-\frac{9}{8}=\pm\frac{23}{8}$ $\therefore x=\frac{9\pm23}{8}=4$ or $-\frac{7}{4}$.

(14.) $x^2-4ax=-7a^2$. Completing the square,

$x^2-4ax+4a^2=-3a^2$.

Extracting the root, $x-2a=a\sqrt{-3}$ $\therefore x=(2\pm\sqrt{-3})a$.

(15.) $\frac{10}{x}-\frac{14-2x}{x^2}=\frac{22}{9}$. Multiplying by x^2,

$10x-14+2x=\frac{22x^2}{9}$.

Transposing, $\frac{22x^2}{9}-12x=-14$. Multiplying by $\frac{9}{22}$,

$x^2-\frac{54}{11}x=-\frac{63}{11}$. Completing the square,

$x^2-\frac{54}{11}x+\left(\frac{27}{11}\right)^2=\left(\frac{27}{11}\right)^2-\frac{63}{11}=\frac{36}{121}$.

Extracting the root, $x-\frac{27}{11}=\pm\frac{6}{11}$ $\therefore x=\frac{27\pm6}{11}=3$ or $\frac{21}{11}$.

(16.) $x+\sqrt{(5x+10)}=8$. Transposing, $\sqrt{(5x+10)}=8-x$.
Squaring, $5x+10=64-16x+x^2$. Transposing,
$x^2-21x=-54$. Completing the square,

$x^2-21x+\left(\frac{21}{2}\right)^2=\frac{441}{4}-54=\frac{225}{4}$.

Extracting the root, $x-\frac{21}{2}=\pm\frac{15}{2}$ $\therefore x=\frac{21\pm15}{2}=18$ or 3.

(17.) $\frac{3x+4}{5}-\frac{30-2x}{x-6}=\frac{7x-14}{10}$. Adding $\frac{7}{5}$ to each side,

$$\frac{3x+11}{5}-\frac{30-2x}{x-6}=\frac{7x}{10}.$$

Or, subtracting $\frac{6x}{10}$,

$$\frac{11}{5}-\frac{30-2x}{x-6}=\frac{x}{10}.$$

Multiplying by $10(x-6)$, $22x-132-300+20x=x^2-6x$.

Transposing, $x^2-48x=-432$.

Completing the square, $x^2-48x+24^2=576-432=144$.

Extracting the root, $x-24=\pm 12$ $\therefore$ $x=24\pm 12=36$ or 12.

(18.) $x+\sqrt{(10x+6)}=9$. Transposing, $\sqrt{(10x+6)}=9-x$.

Squaring, $10x+6=81-18x+x^2$.

Transposing, $x^2-28x=-75$.

Completing the square, $x^2-28x+14^2=196-75=121$.

Extracting the root, $x-14=\pm 11$ $\therefore$ $x=14\pm 11=25$ or 3.

(19.) $(x+2)^2=2x^2+8$; that is, $x^2+4x+4=2x^2+8$.

Transposing, $x^2-4x+4=0$; that is, $(x-2)^2=0$ $\therefore$ $x-2=0$ $\therefore$ $x=2$.

(20.) $\frac{x+22}{3}-\frac{9x-6}{2}=\frac{4}{x}$. Multiplying by $6x$,

$2x^2+44x-27x^2+18x=24$. Transposing, $25x^2-62x=-24$.

Dividing by 25, $x^2-\frac{62}{25}x=-\frac{24}{25}$. Completing the square,

$$x^2-\frac{62}{25}x+\left(\frac{31}{25}\right)^2=\frac{961}{625}-\frac{24}{25}=\frac{361}{625}.$$

Extracting the root, $x-\frac{31}{25}=\pm\frac{19}{25}$ $\therefore$ $x=\frac{31\pm 19}{25}=2$ or $\frac{12}{25}$.

(21.) $\frac{2x}{9}-2=\frac{3x-16}{18}-\frac{4x-3}{4x+3}$, or $\frac{2x}{9}-\frac{18}{9}=\frac{x}{6}-\frac{8}{9}-\frac{4x-3}{4x+3}$.

Transposing, $\frac{2x-10}{9}=\frac{x}{6}-\frac{4x-3}{4x+3}$. Multiplying by $18(4x+3)$,

$$2(2x-10)(4x+3)=3x(4x+3)-18(4x-3);$$

that is, $16x^2-68x-60=12x^2+9x-72x+54$.

Transposing, $4x^2-5x=114$. Dividing by 4, $x^2-\frac{5}{4}x=\frac{57}{2}$.

Completing the square, $x^2-\frac{5}{4}x+\left(\frac{5}{8}\right)^2=\frac{57}{2}+\frac{25}{8^2}=\frac{1849}{8^2}$.

Extracting the root, $x-\frac{5}{8}=\pm\frac{43}{8}$ $\therefore x=\frac{5\pm43}{8}=6$ or $-4\frac{3}{4}$.

(22.) $\frac{x-3}{x+5}-\frac{x+4}{x-7}=2\frac{7}{9}$. Clearing fractions,

$9(x-7)(x-3)-9(x+5)(x+4)=25(x+5)(x-7)$; that is,

$9(x^2-10x+21-x^2-9x-20)=25(x^2-2x-35)$,

or $9(1-19x)=25(x^2-2x-35)$.

Transposing, $25x^2+121x=884$. Dividing by 25,

$x^2+\frac{121}{25}x=\frac{884}{25}$. Completing the square,

$$x^2+\frac{121}{25}x+\left(\frac{121}{50}\right)^2=\frac{884}{25}+\frac{14641}{50^2}=\frac{103041}{50^2}.$$

Extracting the root,

$$x+\frac{121}{50}=\pm\frac{321}{50}\ \therefore x=\frac{-121\pm321}{50}=4 \text{ or } -8\frac{21}{25}.$$

(23.) $x^2-(a+b)x+ab=0$ $\therefore x^2-(a+b)x=-ab$.

Completing the square, $x^2-(a+b)x+\frac{(a+b)^2}{4}=\frac{(a-b)^2}{4}$.

Extracting the root,

$$x-\frac{a+b}{2}=\pm\frac{a-b}{2}\ \therefore x=\frac{a+b\pm(a-b)}{2}=a \text{ or } b.$$

NOTE.—As observed in the Note at p. 56, we can always pronounce at once on the roots of an equation, whenever the simple factors of the first member, equated to zero, are known. It is obvious, that in the present example the factors of $x^2-(a+b)x+ab$ are $(x-a)(x-b)$, so that the roots are at once seen to be $x=a$ and $x=b$. The sum of the roots, with changed signs, is always equal to the coefficient of x, and their product equal to the third term. (See Algebra, p. 66).

Many quadratics may be solved at sight by means of this principle. Thus: ex. 1, p. 73, Algebra, is $x^2-8x-9=0$,

where it is immediately seen that $-8=-9+1$, and that $-9=-9\times1$: hence, changing the signs of -9 and 1, the roots are $x=9$ and $x=-1$. Again: ex. 3 is $x^2-3x-10=0$, where $-3=-5+2$, and $-10=-5\times2$: therefore, the roots are $x=5$, and $x=-2$, and so in other instances.

(24.) $4x+4\sqrt{(x+2)}=7 \therefore 4\sqrt{(x+2)}=7-4x$.
Squaring, $16(x+2)=(7-4x)^2 \therefore 16x+32=49-56x+16x^2$
$\therefore 16x^2-72x=-17$.

Dividing by 16, $x^2-\frac{9}{2}x=-\frac{17}{16}$.

Completing the square, $x^2-\frac{9}{2}x+\frac{81}{16}=\frac{81}{16}-\frac{17}{16}=\frac{64}{16}$.

Extracting the root, $x-\frac{9}{4}=\pm\frac{8}{4} \therefore x=\frac{9\pm8}{4}=4\frac{1}{4}$ or $\frac{1}{4}$.

(25.) $x=\frac{x-9}{x^{\frac{1}{2}}+3}+15$: that is, $x=x^{\frac{1}{2}}-3+15 \therefore x-x^{\frac{1}{2}}=12$

Completing the square, $x-x^{\frac{1}{2}}+\frac{1}{4}=12\frac{1}{4}=\frac{49}{4}$.

Extracting the root,

$x^{\frac{1}{2}}-\frac{1}{2}=\pm\frac{7}{2} \therefore x_{\frac{1}{2}}=\frac{1\pm7}{2}=-3$ or $4 \therefore x=9$ or 16.

(26.) $\sqrt{(x+6)}+\sqrt{(x+3)}=3\sqrt{x}$.
Squaring, $x+6+2\sqrt{(x+6)(x+3)}+x+3=9x$.
Transposing, $2\sqrt{(x+6)(x+3)}=7x-9$. Squaring, $4(x+6)(x+3)=(7x-9)^2$; that is, $4(x^2+9x+18)=49x^2-126x+81$. Transposing, $45x^2-162x=-9$.

Dividing by 45, $x^2-\frac{18}{5}x=-\frac{1}{5}$.

Completing the square, $x^2-\frac{18}{5}x+\frac{81}{25}=\frac{81}{25}-\frac{1}{5}=\frac{76}{5^2}$.

Extracting the root, $x-\frac{9}{5}=\sqrt{\frac{76}{5^2}} \therefore x=\frac{9\pm\sqrt{76}}{5}$.

(27.) $\frac{\sqrt{(4x+20)}}{4+\sqrt{x}}=\frac{4-\sqrt{x}}{\sqrt{x}}$. Clearing fractions,
$\sqrt{(4x^2+20x)}=16-x$.

Squaring, $4x^2+20x=256-32x+x^2$ $\therefore$ $3x^2+52x=256$.

Dividing by 3, $x^2+\frac{52}{3}x=\frac{256}{3}$. Completing the square,

$$x^2+\frac{52}{3}x+\left(\frac{26}{3}\right)^2=\frac{256}{3}+\frac{676}{9}=\frac{1444}{9}.$$

Extracting the root,

$$x+\frac{26}{3}=\pm\frac{38}{3} \therefore x=\frac{-26\pm38}{3}=4 \text{ or } -\frac{64}{3}.$$

(28.) $\sqrt{x}+2=\sqrt{(7+2x)}$. Squaring, $x+4\sqrt{x}+4=7+2x$. Transposing, $4\sqrt{x}=x+3$. Squaring, $16x=x^2+6x+9$. Transposing, $x^2-10x=-9$. Completing the square,

$$x^2-10x+25=16.$$

Extracting the root, $x-5=\pm4$ $\therefore$ $x=5\pm4=9$ or 1.

(29.) $\frac{x}{x+1}+\frac{x+1}{x}=2\frac{1}{6}$. Clearing fractions,

$$6x^2+6(x+1)^2=13x(x+1);$$

that is, $6x^2+6x^2+12x+6=13x^2+13x$. Transposing,

$x^2+x=6$. Completing the square, $x^2+x+\frac{1}{4}=\frac{25}{4}$.

Extracting the root, $x+\frac{1}{2}=\pm\frac{5}{2}$ $\therefore$ $x=\frac{-1\pm5}{2}=2$ or -3.

(30.) $\frac{4x^2}{3}=\frac{x}{3}+11$ $\therefore 4x^2=x+33$ $\therefore 4x^2-x=33$ $\therefore x^2-\frac{1}{4}x=\frac{33}{4}$.

Completing the square, $x^2-\frac{1}{4}x+\frac{1}{64}=\frac{33}{4}+\frac{1}{64}=\frac{529}{64}$.

Extracting the root, $x-\frac{1}{8}=\pm\frac{23}{8}$ $\therefore$ $x=\frac{1\pm23}{8}=3$ or $-2\frac{3}{4}$.

(31.) $\sqrt{(x-a)}+\sqrt{(x+b)}=2\sqrt{x}$. Squaring, $x-a+2\sqrt{(x-a)(x+b)}+x+b=4x$. Transposing, $2\sqrt{(x-a)(x+b)}=2x+a-b$. Squaring, $4(x-a)(x+b)=(2x+a-b)^2$ $\therefore$ $4(b-a)x-4ab=4(a-b)x+(a-b)^2$
Transposing, $8(b-a)x=(a-b)^2+4ab=(a+b)^2$

$$\therefore x=\frac{(a+b)^2}{8(b-a)}.$$

(32.) $\frac{\sqrt{(a^2x+b)}}{a+\sqrt{x}}=\frac{a-\sqrt{x}}{\sqrt{x}}$. Clearing fractions,

$\sqrt{(a^2x^2+bx)}=a^2-x$. Squaring, $a^2x^2+bx=a^4-2a^2x+x^2$

Transposing, $(a^2-1)x^2+(2a^2+b)x=a^4$. Dividing by a^2-1,

$x^2+\frac{2a^2+b}{a^2-1}x=\frac{a^4}{a^2-1}$. Completing the square,

$$x^2+\frac{2a^2+b}{a^2-1}x+\left(\frac{2a^2+b}{2a^2-2}\right)^2=\frac{4a^4(a^2-1)+(2a^2+b)^2}{4(a^2-1)^2}.$$

Extracting the root, $x+\frac{2a^2+b}{2(a^2-1)}=\frac{\sqrt{(4a^6+4a^2b+b^2)}}{2(a^2-1)}$

$$\therefore x=\frac{-(2a^2+b)\pm\sqrt{(4a^6+4a^2b+b^2)}}{2(a^2-1)}.$$

(33.) $\frac{x+4}{3}-\frac{4x+7}{9}=\frac{7-x}{x-3}-1$ or $\frac{3x+12-4x-7}{9}=\frac{7-x}{x-3}-1$;

that is, $\frac{5-x}{9}=\frac{7-x}{x-3}-1$. Clearing, $(5-x)(x-3)=$

$63-9x-9x+27$.

that is, $-x^2+8x-15=90-18x$ $\therefore$ $x^2-26x=-105$.

Completing the square, $x^2-26x+13^2=169-105=64$.

Extracting the root, $x-13=\pm 8$ $\therefore$ $x=13\pm 8=21$ or 5.

(34.) $\sqrt{(5a+x)}+\sqrt{(5a-x)}=\frac{12a}{\sqrt{(5a+x)}}$. Clearing,

$5a+x+\sqrt{(25a^2-x^2)}=12a$. Transposing,

$\sqrt{(25a^2-x^2)}=7a-x$. Squaring, $25a^2-x^2=49a^2-14ax+x^2$.

Transposing, $2x^2-14ax=-24a^2$ $\therefore$ $x^2-7ax=-12a^2$.

Completing the square, $x^2-7ax+\frac{49a^2}{4}=\frac{49a^2}{4}-12a^2=\frac{a^4}{4}$.

Extracting the root, $x-\frac{7a}{2}=\pm\frac{a}{2}$ $\therefore$ $x=\frac{7a\pm a}{2}=4a$ or $3a$.

(35.) $x^4-8x^2=9$ or $x^4-8x^2-9=0$. Here it is obvious that $-8=-9+1$, and that $-9=-9\times 1$: hence (NOTE, p. 60), $x^2=9$ or -1 $\therefore$ $x=\pm 3$ or $\sqrt{-1}$.

Otherwise.—Completing the square, $x^4-8x^2+16=25$.

Extracting the root, $x^2-4=\pm 5$ $\therefore$ $x^2=4\pm 5=9$ or -1

$\therefore$ $x=\pm 3$ or $\sqrt{-1}$.

(36.) $x^6-4x^3=32$. Completing the square,
$x^6-4x^3+4=36$. Extracting the root,
$x^3-2=\pm 6 \therefore x^3=2\pm 6=8$ or $-4 \therefore x=2$ or $\sqrt[3]{-4}$.

(37.) $\dfrac{nx+b}{\sqrt{x}}=\dfrac{na+b}{\sqrt{a}}$. Squaring, $\dfrac{(nx+b)^2}{x}=\dfrac{(na+b)^2}{a}$

$\therefore a(nx+b)^2=(na+b)^2x$;

that is, $an^2x^2+2abnx+ab^2=(a^2n^2+2abn+b^2)x$.

Transposing, $an^2x^2-(a^2n^2+b^2)x=-ab^2$. Dividing by an^2,

$$x^2-\frac{a^2n^2+b^2}{an^2}x=-\frac{b^2}{n^2}.$$ Completing the square,

$$x^2-\frac{a^2n^2+b^2}{an^2}x+\left(\frac{a^2n^2+b^2}{2an^2}\right)^2=\left(\frac{a^2n^2+b^2}{2an^2}\right)^2-\frac{b^2}{n^2}=\left(\frac{a^2n^2-b^2}{2an^2}\right)^2.$$

Extracting the root, $x-\dfrac{a^2n^2+b^2}{2an^2}=\pm\dfrac{a^2n^2-b^2}{2an^2}$

$$\therefore x=\frac{a^2n^2+b^2\pm(a^2n^2-b^2)}{2an^2}=a \text{ or } \frac{b^2}{an^2}.$$

NOTE.—If in the given equation a be put for x, the equation will be satisfied: hence, $x=a$ is one of the roots. Consequently, if the equation be freed from radicals, and all the terms be brought to one side, that side will be divisible by $x-a$, and the quotient, equated to 0, will give the other root (see Note, p. 10); or, more simply, if the third term, after rendering the coefficient of x^2 unity, be divided by a, the quotient will be the other root. As shown above, this third term will be $\dfrac{b^2}{n^2}$: hence the other root is $\dfrac{b^2}{an^2}$, as confirmed by the more lengthy operation above.

(38.) $x^4-2x^2=3$, or $x^4-2x^2-3=0$. Here it is easy to see that $-2=-3+1$, and that $-3=-3\times 1$: hence (Note, p. 10), $x^2=3$, or $-1 \therefore x=\sqrt{3}$, or $\sqrt{-1}$.

Otherwise.—Completing the square, $x^4-2x^2+1=4$.

Extracting the root, $x^2-1=\pm 2 \therefore x^2=1\pm 2=3$ or -1

$\therefore x=\sqrt{3}$, or $\sqrt{-1}$.

(39.) $\sqrt{(4a+x)}+\sqrt{(a+x)}=2\sqrt{(2a+x)}$.

Squaring, $4a+x+2\sqrt{(4a+x)(a+x)}+a+x=8a+4x$.

Transposing, $2\sqrt{(4a+x)(a+x)}=2x+3a$.

Squaring, $4(x^2+5ax+4a^2)=4x^2+12ax+9a^2$.
Transposing, $8ax=-7a^2$ $\therefore$ $x=-\frac{7}{8}a$.

(40.) $x^3-x^{\frac{3}{2}}=56$.

Completing the square, $x^3-x^{\frac{3}{2}}+\frac{1}{4}=56\frac{1}{4}=\frac{225}{4}$.

Extracting the root, $x^{\frac{3}{2}}-\frac{1}{2}=\pm\frac{15}{2}$ $\therefore$ $x^{\frac{3}{2}}=\frac{1\pm15}{2}=8$, or -7
$\therefore$ $x^3=64$, or 49 $\therefore$ $x=4$, or $\sqrt[3]{49}$.

(41.) $x+5=\sqrt{(x+5)}+6$. Transposing, $(x+5)-(x+5)^{\frac{1}{2}}=6$.

Completing the square, $(x+5)-(x+5)^{\frac{1}{2}}+\frac{1}{4}=6\frac{1}{4}=\frac{25}{4}$.

Extracting the root, $(x+5)^{\frac{1}{2}}-\frac{1}{2}=\pm\frac{5}{2}$ $\therefore$ $(x+5)^{\frac{1}{2}}=\frac{1\pm5}{2}$
$=3$ or -2 $\therefore$ $x+5=9$ or 4 $\therefore$ $x=4$ or -1.

(42.) $\sqrt{(2x+1)}+2\sqrt{x}=\frac{21}{\sqrt{(2x+1)}}$.

Clearing, $2x+1+2\sqrt{(2x^2+x)}=21$.
Transposing, $2\sqrt{(2x^2+x)}=20-2x$ $\therefore$ $\sqrt{(2x^2+x)}=10-x$.
Squaring, $2x^2+x=100-20x+x^2$.
Transposing, $x^2+21x=100$.

Completing the square, $x^2+21x+\left(\frac{21}{2}\right)^2=\frac{841}{4}$.

Extracting the root,

$$x+\frac{21}{2}=\pm\frac{29}{2} \therefore x=\frac{-21\pm29}{2}=4, \text{ or } -25.$$

(43.) $x^6+20x^3=69$.

Completing the square, $x^6+20x^3+100=169$.
Extracting the root, $x^3+10=\pm13$ $\therefore$ $x^3=-10\pm13=3$ or -23
$\therefore$ $x=\sqrt[3]{3}$, or $\sqrt[3]{-23}$.

(44.) $\frac{a+x}{\sqrt{(a-x)}}+\frac{a-x}{\sqrt{(a+x)}}=2\sqrt{a}$. Clearing fractions,

$$(a+x)^{\frac{3}{2}}+(a-x)^{\frac{3}{2}}=2(a^2-x^2)^{\frac{1}{2}}\sqrt{a}.$$

Squaring, $(a+x)^3+2(a^2-x^2)^{\frac{3}{2}}+(a-x)^3=4a(a^2-x^2)$;

that is, $2a^3+6ax^2+2(a^2-x^2)^{\frac{3}{2}}=4a^3-4ax^2$

$\therefore (a^2-x^2)^{\frac{3}{2}}=a^3-5ax^2$.

Squaring, $(a^2-x^2)^3=a^6-10a^4x^2+25a^2x^4$;

that is, $a^6-3a^4x^2+3a^2x^4-x^6=a^6-10a^4x^2+25a^2x^4$.

Transposing, $x^6+22a^2x^4-7a^4x^2=0 \therefore x^4+22a^2x^2=7a^4$.

Completing the square, $x^4+22a^2x^2+121a^4=128a^4$.

Extracting the root, $x^2+11a^2=a^2\surd(64\times 2)=8a^2\surd 2$

$\therefore x^2=a^2(8\surd 2-11) \therefore x=\pm a\surd(8\surd 2-11)$.

(45.) $x\sqrt{\left(\frac{a}{x}-1\right)}=\surd(x^2-b^2)$.

Squaring, $x^2\left(\frac{a}{x}-1\right)=x^2-b^2$, or $ax-x^2=x^2-b^2$.

Transposing, and dividing by 2, $x^2-\frac{a}{2}x=\frac{b^2}{2}$.

Completing the square, $x^2-\frac{a}{2}x+\frac{a^2}{16}=\frac{a^2+8b^2}{16}$.

Extracting the root, $x-\frac{a}{4}=\frac{\surd(a^2+8b^2)}{4}$.

Transposing, $x=\frac{1}{4}\{a\pm\surd(a^2+8b^2)\}$.

(46.) $(a+1)(x-1)^2=2(x^2+1)$;

that is, $(a+1)x^2-2(a+1)x+a+1=2x^2+2$. Transposing,

$(a-1)x^2-2(a+1)x=1-a \therefore x^2-\frac{2(a+1)}{a-1}x=\frac{1-a}{a-1}=-1$.

Completing the square,

$$x^2-\frac{2(a+1)}{a-1}x+\left(\frac{a+1}{a-1}\right)^2=\left(\frac{a+1}{a-1}\right)^2-1=\frac{4a}{(a-1)^2}.$$

Extracting the root,

$$x-\frac{a+1}{a-1}=\frac{2\surd a}{a-1} \therefore x=\frac{a\pm 2\surd a+1}{a-1}=\frac{(\surd a\pm 1)^2}{a-1}.$$

Dividing numerator and denominator by $\surd a\pm 1$, we have

$$x=\frac{\surd a+1}{\surd a-1}, \text{ or } \frac{\surd a-1}{\surd a+1}.$$

(47.) $x^2-7x+\surd(x^2-7x+18)=24$. Add 18 to both sides,

then $(x^2-7x+18)+\surd(x^2-7x+18)=42$.

Completing the square,

$$(x^2-7x+18)+\sqrt{(x^2-7x+18)}+\tfrac{1}{4}=42\tfrac{1}{4}=\frac{169}{4}.$$

Extracting the root, $\sqrt{(x^2-7x+18)}+\tfrac{1}{2}=\pm\frac{13}{2}$

$$\therefore \sqrt{(x^2-7x+18)}=\frac{-1\pm 13}{2}=6 \text{ or } -7.$$

Squaring, $x^2-7x+18=36$ or 49

$\therefore x^2-7x=18$ or $x^2-7x=31$. Completing the square,

$$x^2-7x+\frac{49}{4}=\frac{121}{4}, \text{ or } x^2-7x+\frac{49}{4}=\frac{173}{4}.$$

Extracting the root, $x-\frac{7}{2}=\pm\frac{11}{2}$, or $x-\frac{7}{2}=\frac{\sqrt{173}}{2}$

$$\therefore x=\frac{7\pm 11}{2}=9 \text{ or } -2; \text{ or } x=\frac{7\pm\sqrt{173}}{2}.$$

$$(48.)\ \left(\frac{1}{x}\right)^3+a=\left(\frac{1}{x}\right)^{\frac{3}{2}} \therefore \left(\frac{1}{x}\right)^3-\left(\frac{1}{x}\right)^{\frac{3}{2}}=-a.$$

Completing the square, $\left(\frac{1}{x}\right)^3-\left(\frac{1}{x}\right)^{\frac{3}{2}}+\frac{1}{4}=\frac{1}{4}-a=\frac{1-4a}{4}.$

Extracting the root,

$$\left(\frac{1}{x}\right)-\tfrac{1}{2}=\frac{\sqrt{(1-4a)}}{2} \therefore \left(\frac{1}{x}\right)^{\frac{3}{2}}=\frac{1\pm\sqrt{(1-4a)}}{2}.$$

Or, multiplying numerator and denominator by $1\mp\sqrt{(1-4a)}$,

$$\left(\frac{1}{x}\right)^{\frac{3}{2}}=\frac{2a}{1\mp\sqrt{(1-4a)}} \therefore \text{reversing the fractions,}$$

$$x=\left\{\frac{1\mp\sqrt{(1-4a)}}{2a}\right\}^{\frac{2}{3}}.$$

$$(49.)\ 5x+\frac{125}{5^x}=30 \therefore 5^{2x}-30\,.\,5^x=-125,$$

Completing the square, $5^{2x}-30\,.\,5^x+225=100$.

Extracting the root, $5^x-15=\pm 10 \therefore 5^x=15\pm 10=25$ or 5.

Consequently, x must be either 2 or 1.

$$(50.)\ \sqrt{(a+x)}+\sqrt{(a-x)}=\frac{b}{\sqrt{(a+x)}}.$$

Multiplying by $\sqrt{(a+x)}$,

$$a+x+\sqrt{(a^2-x^2)}=b \therefore \sqrt{(a^2-x^2)}=b-a-x.$$

Squaring, $a^2-x^2=(b-a)^2-2(b-a)x+x^2$.

Transposing, $2x^2-2(b-a)x=2ab-b^2$.

Dividing by 2, and completing the square,

$$x^2-(b-a)x+\left(\frac{b-a}{2}\right)^2=\frac{(b-a)^2+4ab-2b^2}{4}=\frac{a^2+2ab-b^2}{4}.$$

Extracting the root,

$$x-\frac{b-a}{2}=\frac{\sqrt{(a^2+2ab-b^2)}}{2} \therefore x=\tfrac{1}{2}\{b-a\pm\sqrt{(a^2+2ab-b^2)}\}.$$

(51.) $x^2-2x+6\sqrt{(x^2-2x+5)}=11$. Add 5 to each side, then $(x^2-2x+5)+6\sqrt{(x^2-2x+5)}=16$.

Completing the square,

$$(x^2-2x+5)+6\sqrt{(x^2-2x+5)}+9=25.$$

Extracting the root,

$$\sqrt{(x^2-2x+5)}+3=\pm 5 \therefore \sqrt{(x^2-2x+5)}=2 \text{ or } -8.$$

Squaring, $x^2-2x+5=4$, or $x^2-2x+5=64$

$\therefore x^2-2x=-1$, or $x^2-2x=59$.

Completing the square, $x^2-2x+1=0$, or $x^2-2x+1=60$.

Extracting the root,

$$x-1=0 \therefore x=1, \text{ or } x-1=2\sqrt{15} \therefore x=1\pm 2\sqrt{15}.$$

(52.) $x^2(x-1)=8(x+2)$, or $x^3-x^2-8x-16=0$. This is a cubic equation; but, by means of an algebraical artifice, it can be readily solved by quadratics. Instances of reducing cubics to quadratics will be found in the Algebra, pp. 69, 73, &c. The reduction generally requires some ingenuity, and is not to be effected by prescribed rules: the artifice in the present example is as follows:—

From the given equation, $x^3-(x+4)^2=0$

Subtract $16x-16x \quad =0$

There remains $x(x^2-16)-(x-4)^2=0$

Hence the equation is the same as

$$(x-4)\{x(x+4)-(x-4)\}=0.$$

And as an expression is rendered 0 by equating either of its factors to 0, this equation supplies the two equations,

$$x-4=0 \text{ and } x^2+3x+4=0.$$

From the first of these, we find that $x=4$ is one of the values of x; and completing the square in the second, we have

$$x^2+3x+\frac{9}{4}=\frac{9}{4}-4=-\frac{7}{4}.$$

Extracting the root, $x+\frac{3}{2}=\frac{\sqrt{-7}}{2}$ $\therefore x=\frac{-3\pm\sqrt{-7}}{2}$.

These are the two values of x given in the book; but, besides these imaginary values, the equation has a real value, as shown above; namely, the value $x=4$: this value, substituted in the proposed equation, makes each side of it 48.

(53.) $x^4-2x^3+x=a$. The solution of this equation, like that of the preceding, depends upon an algebraical artifice.

$$\begin{array}{lll} \text{To the given equation,} & x^4-2x^3+x & =a \\ \text{Add} & x^2-x^2 & =0 \\ \hline \text{There results} & (x^2-x)^2-(x^2-x) & =a. \end{array}$$

For convenience, put y for x^2-x, then the equation is $y^2-y=a$.

Completing the square, $y^2-y+\frac{1}{4}=\frac{4a+1}{4}$.

Extracting the root, $y-\frac{1}{2}=\frac{\sqrt{(4a+1)}}{2}$ $\therefore y=\frac{1\pm\sqrt{(4a+1)}}{2}$.

Consequently, restoring the value of y,

$$x^2-x=\frac{1\pm\sqrt{(4a+1)}}{2}.$$

Completing the square, $x^2-x+\frac{1}{4}=\frac{3\pm2\sqrt{(4a+1)}}{4}$.

Extracting the root,

$$x-\frac{1}{2}=\frac{\sqrt{\{3\pm2\sqrt{(4a+1)}\}}}{2} \therefore x=\frac{1\pm\sqrt{\{3\pm2\sqrt{(4a+1)}\}}}{2}.$$

NOTE.—As the first member of the equation $y^2-y=a$ is the very same in form as the first member of the equation just solved, it would have been sufficient, for the solution of the latter, if we had merely substituted $\frac{1\pm\sqrt{(4a+1)}}{2}$ for a in the solution of the former.

(54.) $\frac{x}{\sqrt{x}+\sqrt{(a-x)}}+\frac{x}{\sqrt{x}-\sqrt{(a-x)}}=\frac{b}{\sqrt{x}}$.

Clearing fractions,

$$x\sqrt{x}\{\sqrt{x}-\sqrt{(a-x)}\}+x\sqrt{x}\{\sqrt{x}+\sqrt{(a-x)}\}=b(x-a+x)$$

that is, $2x^2=2bx-ab$ ∴ $x^2-bx=-\frac{ab}{2}$.

Completing the square, $x^2-bx+\frac{b^2}{4}=\frac{b^2-2ab}{4}$.

Extracting the root, $x-\frac{b}{2}=\frac{\sqrt{(b^2-2ab)}}{2}$ ∴ $x=\frac{b\pm\sqrt{(b^2-2ab)}}{2}$

(55.) $x^{-3}+\frac{1}{x\sqrt{x}}=2$. Multiplying by $x^{\frac{3}{2}}$, $x^{-\frac{3}{2}}+1=2x^{\frac{3}{2}}$.

Put y for $x^{\frac{3}{2}}$, then the equation is $\frac{1}{y}+1=2y$ ∴ $2y^2-y=1$

∴ $y^2-\frac{1}{2}y=\frac{1}{2}$. Completing the square, $y^2-\frac{1}{2}y+\frac{1}{16}=\frac{9}{16}$.

Extracting the root, $y-\frac{1}{4}=\pm\frac{3}{4}$ ∴ $y=\frac{1\pm 3}{4}=1$ or $-\frac{1}{2}$.

Restoring the value of y, $x^{\frac{3}{2}}=1$ or $x^{\frac{3}{2}}=-\frac{1}{2}$

∴ $x=1^{\frac{2}{3}}=1$ or $x=(-\frac{1}{2})^{\frac{2}{3}}=\frac{1}{\sqrt[3]{4}}=\frac{\sqrt[3]{2}}{\sqrt[3]{8}}=\frac{\sqrt[3]{2}}{2}$.

(56.) $ax-b^2=x^2-2b\sqrt{(a^2-ax+x^2)}$.

Transposing, $x^2-ax-2b\sqrt{(a^2-ax+x^2)}=-b^2$.

Adding a^2 to each side,

$(a^2-ax+x^2)-2b\sqrt{(a^2-ax+x^2)}=a^2-b^2$;

or, putting y^2 for the first term, $y^2-2by=a^2-b^2$.

Completing the square, $y^2-2by+b^2=a^2$

Extracting the root, $y-b=\pm a$ ∴ $y=b\pm a$.

Restoring the value of y^2,

$x^2-ax+a^2=(b\pm a)^2$ ∴ $x^2-ax=b^2\pm 2ab$

Completing the square, $x^2-ax+\frac{a^2}{4}=\frac{a^2\pm 8ab+4b^2}{4}$.

Extracting the root, $x-\frac{a}{2}=\frac{\sqrt{(a^2\pm 8ab+4b^2)}}{2}$

$$\therefore x=\frac{a\pm\sqrt{(a^2\pm 8ab+4b^2)}}{2}.$$

(57.) $9x-3x^2+4\sqrt{(x^2-3x+5)}=11$.

Dividing by -3, $x^2-3x-\frac{4}{3}\sqrt{(x^2-3x+5)}=-\frac{11}{3}$.

Adding 5 to each side, $(x^2-3x+5)-\frac{4}{3}\sqrt{(x^2-3x+5)}=\frac{4}{3}$.

or, putting y^2 for the first term, $y^2-\frac{4}{3}y=\frac{4}{3}$.

Completing the square,

$$y^2-\frac{4}{3}y+\frac{4}{9}=\frac{16}{9}\ \therefore\ y-\frac{2}{3}=\pm\frac{4}{3}\ \therefore\ y=\frac{2\pm4}{3}=2 \text{ or } -\frac{2}{3}.$$

Restoring the value of y^2,

$$x^2-3x+5=4, \text{ or } \frac{4}{9}\ \therefore\ x^2-3x=-1, \text{ or } -\frac{41}{9}.$$

Completing the square, $x^2-3x+\frac{9}{4}=\frac{5}{4}$, or $-\frac{83}{16}$.

Extracting the root,

$$x-\frac{3}{2}=\frac{\sqrt{5}}{2}, \text{or } \frac{\sqrt{-83}}{6}\ \therefore\ x=\frac{3\pm\sqrt{5}}{3}, \text{or } \frac{9\pm\sqrt{-83}}{6}.$$

(58.) $\sqrt{\frac{a}{x}}+\sqrt{\frac{x}{a}}=\frac{\sqrt{(2ax-x^2)}}{x}=\frac{\sqrt{(2a-x)}}{\sqrt{x}}$.

Multiplying by $\sqrt{x}$, $\sqrt{a}+\frac{x}{\sqrt{a}}=\sqrt{(2a-x)}$.

Squaring, $a+2x+\frac{x^2}{a}=2a-x$.

Multiplying by a, and transposing, $x^2+3ax=a^2$.

Completing the square, $x^2+3ax+\frac{9a^2}{4}=\frac{13a^2}{4}$.

Extracting the root, $x+\frac{3a}{2}=\pm\frac{a}{2}\sqrt{13}\ \therefore\ x=-\frac{1}{2}a(3\mp\sqrt{13})$.

(59.) $x^2+\frac{1}{x^2}+x+\frac{1}{x}=4$. Add 2 to each side, then

$$\left(x^2+2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)=6; \text{ that is, } \left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)=6.$$

Put the first term $=y^2\ \therefore\ y^2+y=6\ \therefore\ y^2+y+\frac{1}{4}=\frac{25}{4}$.

Extracting the root, $y+\frac{1}{2}=\pm\frac{5}{2}\ \therefore\ y=\frac{-1\pm5}{2}=-3$ or 2.

$$\text{Restoring the value of } y,\ x+\frac{1}{x}=-3 \text{ or } x+\frac{1}{x}=2$$

$$\therefore\ x^2+3x=-1 \text{ or } x-2x=-1.$$

$$\text{Completing squares, } x^2+3x+\frac{9}{4}=\frac{5}{4} \text{ or } x^2-2x+1=0.$$

$$\text{Extracting roots, } x+\frac{3}{2}=\frac{\sqrt{5}}{2}\ \therefore \text{ or } x-1=0$$

$$\therefore\ x=\frac{-3\pm\sqrt{5}}{2} \text{ or } x=1.$$

(60.) $x=\dfrac{12+8x^{\frac{1}{2}}}{x-5}\ \therefore\ x^2-5x-8x^{\frac{1}{2}}-12=0$; or putting y for $x^{\frac{1}{2}}$,

$$y^4-5y^2-8y-12=0.\ldots\ (\text{A}),$$

$$\text{or } y^4-2y^2+4y-3(y^2+4y+4)=0;$$

$$\text{that is, } y^2(y^2-4)+2y(y+2)-3(y+2)^2=0.\ldots\ (\text{B}).$$

This equation is divisible by $y+2$: hence one of the roots of (A) is $y=-2$. Dividing (B) by the factor $y+2$, we have

$$y^2(y-2)+2y-3(y+2)=0;$$

that is, $y^3-2y^2-y-6=0$, or $y^2(y-3)+(y^2-9)-(y-3)=0$. This equation is divisible by $y-3$: hence another root of (A) is $y=3$. Dividing by the factor $y-3$, we have the quadratic

$$y^2+y+3-1=0\ \therefore\ y^2+y=-2.$$

Completing the square,

$$y^2+y+\tfrac{1}{4}=-\frac{7}{4}\ \therefore\ y+\tfrac{1}{2}=\frac{\sqrt{-7}}{2}\ \therefore\ y=\frac{-1\pm\sqrt{-7}}{2}.$$

Restoring the value of y,

$$x^{\frac{1}{2}}=\frac{-1\pm\sqrt{-7}}{2}\ \therefore\ x=\frac{1\pm2\sqrt{-7}-7}{4}=\frac{-3\pm\sqrt{-7}}{2}.$$

And these are the two roots given in the book: but the equation has the two additional roots $x=(-2)^2$ and $x=3^2$.

NOTE.—Although an equation, like that just solved, may have simple integral factors, it is often troublesome to find them; but as the whole number forming the second term of every such factor must be a divisor of the whole number forming the final term of the equation, we may always find that second term by taking for it one or other of the integral factors of the final term of the equation. In the present case, 2 and 3 are the factors of the final term 12, which will be

found to answer, the equation being divisible by $y+2$ and $y-3$.

$$(61.)\quad \sqrt{(x^2+1)}-\sqrt{(x^2-1)}=\frac{x}{\sqrt{(x^4-1)}}.$$

$$\text{Squaring, } 2x^2-2\sqrt{(x^4-1)}=\frac{x^2}{x^4-1}.$$

$$\text{Transposing, } 2\sqrt{(x^4-1)}=2x^2-\frac{x^2}{x^4-1}.$$

$$\text{Squaring, } 4(x^4-1)=4x^4-\frac{4x^4}{x^4-1}+\frac{x^4}{(x^4-1)^2}$$

$$\therefore \frac{x^4}{(x^4-1)^2}-\frac{4x^4}{x^4-1}=-4.$$

$$\text{Clearing fractions, } x^4-4x^4(x^4-1)=-4(x^8-2x^4+1)$$

$$\therefore x^4+4x^4=8x^4-4 \therefore 3x^4=4 \therefore x^2=\frac{2}{\sqrt{3}} \therefore x=\pm\sqrt{\frac{2}{\sqrt{3}}}.$$

NOTE.—Although the double sign is here, and in some other places, prefixed to the *radical*, it is well to apprise the learner that this prefix is superfluous, since the radical sign $\sqrt{}$ implies that the *result* of the operation which it indicates may be either $+$ or $-$, when there is no overruling condition as respects the generation of the quantity under it. In such a case as $\sqrt{(-a)^2}$, or $\sqrt{(+a)^2}$, the result of the operation is, of course, unambiguous, it being $-a$ in the former case, and $+a$ in the latter.

$$(62.)\quad x^3-6x-4=5 \therefore x^3-6x-9=0.$$

This may be written thus: $(x^3-27)-6(x-3)=0$;

or $(x-3)\{(x^2+3x+9)-6\}=0$.

And this equation is satisfied for either

$$x-3=0, \text{ or } x^2+3x+3=0.$$

From the first of these, we infer that one root is $x=3$; the other, by transposing, is $x^2+3x=-3$.

$$\text{Completing the square, } x^2+3x+\frac{9}{4}=-\frac{3}{4}.$$

$$\text{Extracting the root, } x+\frac{3}{2}=\frac{\sqrt{-3}}{2} \therefore x=\frac{-3\pm\sqrt{-3}}{2}.$$

$$\text{Hence the roots are 3, or } \frac{-3\pm\sqrt{-3}}{2}.$$

(63.) $3x^2=\frac{2}{x}+\frac{13}{3}$. Clearing fractions, $9x^3-13x-6=0$, which may be written thus: $x(9x^2-4)-3(3x+2)=0$.

Hence the equation is $(3x+2)\{x(3x-2)-3\}=0$.

which is satisfied for either $3x+2=0$, or $3x^2-2x-3=0$. From the first of these, we infer that one root is $x=-\frac{2}{3}$. The other, by transposing and dividing by 3, is $x^2-\frac{2}{3}x=1$.

Completing the square, $x^2-\frac{2}{3}x+\frac{1}{9}=\frac{10}{9}$.

Extracting the root, $x-\frac{1}{3}=\frac{\sqrt{10}}{3}$ $\therefore x=\frac{1\pm\sqrt{10}}{3}$.

Hence the roots are $x=-\frac{2}{3}$, or $\frac{1\pm\sqrt{10}}{3}$.

(64.) $ax+2\sqrt{(n^2x+nax^2)}=(3x-1)n$.

Transposing, $2\sqrt{(n^2x+nax^2)}=3nx-(n+ax)$.

Squaring, $4nx(n+ax)=9n^2x^2-6nx(n+ax)+(n+ax)^2$.

Transposing, $(n+ax)^2-10nx(n+ax)=-9n^2x^2$.

Dividing by n^2x^2, $\left(\frac{n+ax}{nx}\right)^2-10\left(\frac{n+ax}{nx}\right)=-9$.

Put y for the second fraction, then $y^2-10y=-9$

$\therefore y^2-10y+25=16$.

Extracting the root, $y-5=\pm4$ $\therefore y=1$ or 9

$\therefore \frac{n+ax}{nx}=1$ or 9 $\therefore \frac{1}{x}=1-\frac{a}{n}=\frac{n-a}{n}$, or $9-\frac{a}{n}=\frac{9n-a}{n}$

$\therefore x=\frac{n}{n-a}$, or $\frac{n}{9n-a}$.

(65.) $2\{\sqrt{(1-x)}+1\}=\frac{x}{\sqrt{(1+x)}-1}$.

This equation is the same as $2\{1+\sqrt{(1-x)}\}=\frac{1-(1-x)}{\sqrt{(1+x)}-1}$.

Dividing by $1+\sqrt{(1-x)}$,

$2=\frac{1-\sqrt{(1-x)}}{\sqrt{(1+x)}-1}$ $\therefore 2\sqrt{(1+x)}-2=1-\sqrt{(1-x)}$

$\therefore 2\sqrt{(1+x)}=3-\sqrt{(1-x)}$.

Squaring, $4(1+x)=9-6\sqrt{(1-x)}+1-x$.

Transposing, $6\sqrt{(1-x)}=6-5x$.

Squaring, $36(1-x)=36-60x+25x^2$ $\therefore$ $25x^2-24x=0$

$$\therefore x=\frac{24}{25}.$$

(66.) $(x-b)^{2n}+2b^n x=b^{n+1}(2+b^{n-1})+2(x-b)^{n+1}$.

Transposing, $(x-b)^{2n}-b^{2n}-2(x-b)\{(x-b)^n-b^n\}=0$.

Here it is plain that $(x-b)^n-b^n$ is a factor of the equation: hence (see Note, p. 10),

$$(x-b)^n-b^n=0 \therefore x-b=b \therefore x=2b.$$

This, therefore, is one root of the proposed equation. The roots which render the other factor zero are not determinable.

$$(67.)\ \left(x-\frac{ab}{x}\right)^2=\frac{a^2+ab}{2}\left(\frac{a^2}{x^2}+1\right),$$

or multiplying by x^2, $(x^2-ab)^2=\frac{a}{2}(a+b)(a^2+x^2)$.

Subtract $\frac{a}{2}(a+b)(x^2-ab)$ from each side, then

$$(x^2-ab)^2-\frac{a}{2}(a+b)(x^2-ab)=\frac{a}{2}(a+b)(a^2+ab)=\frac{a^2}{2}(a+b)^2.$$

$$\text{Add}\left(\frac{a(a+b)}{4}\right)^2 \therefore \left\{(x^2-ab)-\frac{a}{4}(a+b)\right\}^2=\frac{9a^2}{16}(a+b)^2$$

$$\therefore x^2-ab-\frac{a(a+b)}{4}=\pm\frac{3a}{4}(a+b)$$

$$\therefore x^2=ab+\frac{a}{4}\{a+b\pm 3(a+b)\}=2ab+a^2 \text{ or } \frac{ab-a^2}{2}$$

$$\therefore x=\sqrt{(a^2+2ab)} \text{ or } \sqrt{\frac{ab-a^2}{2}}.$$

(68.) $x^4=-1$. Add $2x^2+1$ to each side, then

$x^4+2x^2+1=2x^2$ $\therefore$ $x^2+1=\sqrt{2}\,.\,x$ $\therefore$ $x^2-\sqrt{2}\,.\,x=-1$.

Completing the square, $x^2-\sqrt{2}\,.\,x+\frac{1}{2}=-\frac{1}{2}$.

Extracting the root, $x-\sqrt{\frac{1}{2}}=\sqrt{-\frac{1}{2}}$ $\therefore$ $x=\pm\sqrt{\frac{1}{2}}\pm\sqrt{-\frac{1}{2}}$;

that is, $x=\dfrac{\pm 1\pm\sqrt{-1}}{\sqrt{2}}$.

(69.) $x^{4n}-2x^{3n}+x^n=6$. Put y for x^n, then

$y^4-2y^3+y=6$ $\therefore$ $y^4-2y^3=6-y$.

Add y^2 to each side, then $y^4-2y^3+y^2=y^2-y+6$;
that is, $(y^2-y)^2=(y^2-y)+6 \therefore (y^2-y)^2-(y^2-y)=6$.

Completing the square, $(y^2-y)^2-(y^2-y)+\frac{1}{4}=\dfrac{25}{4}$.

Extracting the root, $y^2-y-\frac{1}{2}=\pm\frac{5}{2} \therefore y^2-y=3$, or -2.

Completing the square, $y^2-y+\frac{1}{4}=\dfrac{13}{4}$, or $-\dfrac{7}{4}$.

$$\therefore y-\tfrac{1}{2}=\tfrac{1}{2}\sqrt{13}, \text{ or } \tfrac{1}{2}\sqrt{-7} \therefore y=\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{13}, \text{ or } \tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{-7}$$

$$\therefore \sqrt[n]{y}=x=\sqrt[n]{(\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{13})}, \text{ or } \sqrt[n]{(\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{-7})}.$$

(70.) $\{(x-2)^2-x\}^2-90+x=(x-2)^2$

$$\therefore \{(x-2)^2-x\}^2-\{(x-2)^2-x\}=90.$$

Put y for $(x-2)^2-x$, then the equation is $y^2-y=90$. Completing the square,

$$y^2-y+\tfrac{1}{4}=\frac{361}{4} \therefore y-\tfrac{1}{2}=\pm\frac{19}{2} \therefore y=10, \text{ or } -9$$

$$\therefore (x-2)^2-x=10 \text{ or } -9 \therefore (x-2)^2-(x-2)=12 \text{ or } -7.$$

Completing the square, $(x-2)^2-(x-2)+\frac{1}{4}=\dfrac{49}{4}$ or $-\dfrac{27}{4}$.

$$\therefore x-2-\tfrac{1}{2}=\pm\frac{7}{2} \text{ or } x-2-\tfrac{1}{2}=\frac{3}{2}\sqrt{-3}$$

$$\therefore x=6, \text{ or } -1 \text{ or } \frac{5\pm3\sqrt{-3}}{2}.$$

(71.) $\sqrt[m]{(1+x)^2}-\sqrt[m]{(1-x^2)}=\sqrt[m]{(1-x^2)}$.

Divide by $(\sqrt[m]{1-x})^2$, then $\left(\dfrac{1+x}{1-x}\right)^{\frac{2}{m}}-1=\left(\dfrac{1+x}{1-x}\right)^{\frac{1}{m}}$.

Put this fraction $=y$, then

$$y^2-y=1 \therefore y^2-y+\tfrac{1}{4}=\frac{5}{4} \therefore y-\tfrac{1}{2}=\pm\tfrac{1}{2}\sqrt{5} \therefore y=\frac{1\pm\sqrt{5}}{2}$$

$\therefore y^m=\dfrac{1+x}{1-x}=\left(\dfrac{1\pm\sqrt{5}}{2}\right)^m$. Then, applying the principle established at page 37 of this Key,

$$x=\frac{(1\pm\sqrt{5})^m-2^m}{(1\pm\sqrt{5})^m+2^m}.$$

(72.) $8x^4+4x^3-18x^2+11x-2=0$. Divide by $x+2$, the quotient will be $8x^3-12x^2+6x-1$, which is a complete cube,

namely, $(2x)^3-3(2x)^2+3(2x)-1=(2x-1)^3$ (see Algebra, p. 38). Consequently, since the proposed equation is the same as $(x+2)(2x-1)^3=0$, it is satisfied by either

$$x+2=0, \text{ or } (2x-1)=0 \therefore x=-2 \text{ or } \tfrac{1}{2}.$$

(73.) $4x^2-20-5\left(x+\dfrac{3}{x}\right)=-\dfrac{36}{x^2} \therefore 4\left(x^2+\dfrac{3^2}{x^2}\right)-5\left(x+\dfrac{3}{x}\right)=20$

Add 4×6 to each side in order to make the first term a complete square, and there results

$$4\left(x+\frac{3}{x}\right)^2-5\left(x+\frac{3}{x}\right)=44.$$

Putting y for $x+\dfrac{3}{x}$, and dividing by 4, $y^2-\dfrac{5}{4}y=11.$

Completing the square, $y^2-\dfrac{5}{4}y+\dfrac{25}{64}=11+\dfrac{25}{64}=\dfrac{729}{64}.$

Extracting the root, $y-\dfrac{5}{8}=\pm\dfrac{27}{8} \therefore y=\dfrac{5\pm 27}{8}=4 \text{ or } -\dfrac{11}{4}$

$$\therefore x+\frac{3}{x}=4, \text{ or } x+\frac{3}{x}=-\frac{11}{4}$$

$$\therefore x^2-4x=-3, \text{ or } x^2+\frac{11}{4}x=-3.$$

Completing squares,

$$x^2-4x+4=1, \text{ or } x^2+\frac{11}{4}x+\left(\frac{11}{8}\right)^2=\frac{121}{64}-3=-\frac{71}{64}.$$

Extracting roots, $x-2=\pm 1$, or $x+\dfrac{11}{8}=\dfrac{\sqrt{-71}}{8}$

$$\therefore x=3 \text{ or } 1, \text{ or } \frac{-11\pm\sqrt{-71}}{8}.$$

(74.) $\dfrac{2}{(x-4)^{\frac{3}{2}}}+\dfrac{(x-4)^{\frac{1}{2}}}{2}=\dfrac{17}{4(x-4)^{\frac{1}{2}}}$. Multiply by $4(x-4)^{\frac{3}{2}}$,

then $8+2(x-4)^2=17(x-4) \therefore (x-4)^2-\dfrac{17}{2}(x-4)=-4.$

Completing the square,

$$(x-4)^2-\frac{17}{2}(x-4)+\left(\frac{17}{4}\right)^2=\frac{289}{16}-4=\frac{225}{16}.$$

Extracting the root,

$$x-4-\frac{17}{4}=\pm\frac{15}{4} \therefore x=4+\frac{17\pm 15}{4}=12 \text{ or } 4\tfrac{1}{2}.$$

(75.) $x^4+4x^3+3x^2+4x+1=0$. This is a reciprocal equation (see Appendix to the Algebra, p. 179). Divide by x^2,

$$\text{then } x^2+4x+3+\frac{4}{x}+\frac{1}{x^2}=0;$$

$$\text{that is, } \left(x^2+\frac{1}{x^2}\right)+4\left(x+\frac{1}{x}\right)+3=0.$$

$$\text{Now, since } \left(x+\frac{1}{x}\right)^2=x^2+\frac{1}{x^2}+2 \therefore x^2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)^2-2$$

$$\therefore \left(x+\frac{1}{x}\right)^2+4\left(x+\frac{1}{x}\right)=-1, \text{ or } y^2+4y=-1$$

$$\therefore y^2+4y+4=3 \therefore y+2=\pm\sqrt{3} \therefore y=-2\pm\sqrt{3}$$

$$\therefore x+\frac{1}{x}=-(2-\sqrt{3}), \text{ or } x+\frac{1}{x}=-(2+\sqrt{3})$$

$$\therefore x^2+(2-\sqrt{3})x=-1, \text{ or } x^2+(2+\sqrt{3})x=-1.$$

Completing squares, $x^2+(2-\sqrt{3})x+\left(\frac{2-\sqrt{3}}{2}\right)^2=\frac{3\mp4\sqrt{3}}{4}$

$$\text{or } x^2+(2+\sqrt{3})x+\left(\frac{2+\sqrt{3}}{2}\right)^2=\frac{3\pm4\sqrt{3}}{4}.$$

Extracting the roots,

$$x+\frac{2-\sqrt{3}}{2}=\frac{\sqrt{(3\mp4\sqrt{3})}}{2}, \text{ or } x+\frac{2+\sqrt{3}}{2}=\frac{\sqrt{(3\pm4\sqrt{3})}}{2}$$

$$\therefore x=\frac{-2\pm\sqrt{3}\pm\sqrt{(3\mp4\sqrt{3})}}{2}.$$

(76.) $(a^{2m}+1)(x^{\frac{1}{2}}-1)^2=2(x+1)$,

$$\text{or } (a^{2m}+1)(x-2\sqrt{x}+1)=2x+2$$

$$\therefore (a^{2m}-1)x-2(a^{2m}+1)\sqrt{x}=-(a^{2m}-1)$$

$$\therefore x-2\frac{a^{2m}+1}{a^{2m}-1}\sqrt{x}=-1.$$

Completing the square,

$$x-2\frac{a^{2m}+1}{a^{2m}-1}\sqrt{x}+\left(\frac{a^{2m}+1}{a^{2m}-1}\right)^2=\frac{4a^{2m}}{(a^{2m}-1)^2}.$$

Extracting the root, $\sqrt{x}-\frac{a^{2m}+1}{a^{2m}-1}=\pm\frac{2a^m}{a^{2m}-1}$

$$\therefore \sqrt{x}=\frac{a^{2m}\pm2a^m+1}{a^{2m}-1}=\frac{(a^m\pm1)^2}{a^{2m}-1}=\frac{a^m\pm1}{a^m\mp1} \therefore x=\left(\frac{a^m\pm1}{a^m\mp1}\right)^2.$$

(77.) $(x^2-9)^2-3=11(x^2-2)$ $\therefore$ $(x^2-9)^2-11(x^2-2)=3$.
Add 11×7 to each side, then $(x^2-9)^2-11(x^2-9)=80$;
or putting y for x^2-9, $y^2-11y=80$ $\therefore y^2-11y+\frac{121}{4}=\frac{441}{4}$.

Extracting the root, $y-\frac{11}{2}=\pm\frac{21}{2}$ $\therefore$ $y=\frac{11\pm 21}{2}=16$, or -5

$\therefore$ $x^2=25$, or 4 $\therefore$ $x=\pm 5$, or ± 2.

(78.) $\sqrt{\{x+\sqrt{(2x-1)}\}}-\sqrt{\{x-\sqrt{(2x-1)}\}}=\frac{3}{5}\sqrt{\frac{10x}{x+\sqrt{(2x-1)}}}$.

Squaring, $2x-2(x-1)=\frac{9}{25}\cdot\frac{10x}{x+\sqrt{(2x-1)}}$

$\therefore$ $\frac{50}{9}=\frac{10x}{x+\sqrt{(2x-1)}}$.

Dividing by 10, and reversing,

$$\frac{9}{5}=\frac{x+\sqrt{(2x-1)}}{x} \therefore \frac{9}{5}-1=\frac{\sqrt{(2x-1)}}{x}$$

$\therefore$ $\frac{4}{5}x=\sqrt{(2x-1)}$. Squaring, $\frac{16}{25}x^2=2x-1$

$$\therefore x^2-\frac{50}{16}x=-\frac{25}{16} \therefore x^2-\frac{50}{16}x+\left(\frac{25}{16}\right)^2=\frac{225}{16^2}.$$

Extracting the root, $x-\frac{25}{16}=\pm\frac{15}{16}$ $\therefore$ $x=\frac{25\pm 15}{16}=\frac{5}{2}$, or $\frac{5}{8}$.

(79.) $\frac{a-\sqrt{(2ax-x^2)}}{a+\sqrt{(2ax-x^2)}}=\frac{x}{a-x}$. Applying the principle at p. 37,

$$\frac{a}{\sqrt{(2ax-x^2)}}=\frac{a}{a-2x} \therefore \sqrt{(2ax-x^2)}=a-2x.$$

Squaring, $2ax-x^2=a^2-4ax+4x^2$

$$\therefore 5x^2-6ax=-a^2 \therefore x^2-\frac{6}{5}ax=-\frac{a^2}{5}.$$

Completing the square, $x^2-\frac{6}{5}ax+\frac{9}{25}a^2=\frac{4}{25}a^2$

$$\therefore x-\frac{3}{5}a=\pm\frac{2}{5}a \therefore x=\frac{3\pm 2}{5}a=a, \text{ or } \frac{a}{5}.$$

(80.) $\frac{x^2-18}{x-12}=\frac{4}{x^2}$. Clearing fractions,

$$x^4-18x^2 \quad = \quad 4x-48$$

$$\text{Adding} \quad 4x^2+49=4x^2 \quad +49$$

$$x^4-14x^2+49=4x^2+4x+1$$

that is, $(x^2-7)^2=4(x+\frac{1}{2})^2$ $\therefore$ $x^2-7=\pm 2(x+\frac{1}{2})$. . . . (A)

$$\therefore x^2-2x=8 \text{ or } x^2+2x=6.$$

Completing squares, $x^2-2x+1=9$ or $x^2+2x+1=7$.

Extracting roots, $x-1=\pm 3$, or $x+1=\pm\sqrt{7}$

$$\therefore x=4, \text{ or } -2, \text{ or } x=-1\pm\sqrt{7}.$$

NOTE.—The learner will perceive that we have prefixed the double sign to the right hand member of the equation (A), and not to the left. It would have been incorrect to do otherwise, for this reason: the square root of $(x^2-7)^2$ is exclusively x^2-7, as the correct square root is exhibited by simply expunging the exponent. In like manner, the correct square root of $(x+\frac{1}{2})^2$ is exclusively $x+\frac{1}{2}$: but $\sqrt{4}$ is ambiguously $+2$ or -2, and it is to *this* square root alone that the double sign above applies. Had the 4 presented itself in the example under the form 2^2, or $(-2)^2$, there would have been no ambiguity as to the sign of the root: in the first case it would have been $+$, and in the second case $-$. (See the NOTE at p. 73).

(81.) $x+4+\left(\frac{x+4}{x-4}\right)=\frac{12}{x-4}$. Multiply by $x-4$; that is, by $(x-4)^{\frac{1}{2}}(x-4)^{\frac{1}{2}}$,

then $x^2-16+(x^2-16)^{\frac{1}{2}}=12$, or $y^2+y=12$,

by putting $x^2-16=y^2$

$$\therefore y^2+y+\tfrac{1}{4}=\frac{49}{4} \;\therefore\; y+\tfrac{1}{2}=\pm\frac{7}{2} \;\therefore\; y=3, \text{ or}=-4$$

$$\therefore y^2=x^2-16=9 \text{ or } 16 \;\therefore\; x=\pm 5, \text{ or } \sqrt{(16\times 2)}=4\sqrt{2}.$$

(82.) $(x+119)^{\frac{1}{3}}+(70-x)^{\frac{1}{3}}=9$. Cubing each side (see p. 38, Algebra),

$$(x+119)+(70-x)+3\times 9(x+119)^{\frac{1}{3}}(70-x)^{\frac{1}{3}}=9^3;$$

that is, $189+3\times 9\{(x+119)(70-x)\}^{\frac{1}{3}}=9^3$.

Dividing by 3×9, $7+\{(x+119)(70-x)\}^{\frac{1}{3}}=27$

$\therefore (x+119)(70-x)=20^3=8000$;

that is, $x^2+49x-8330=-8000$ $\therefore x^2+49x=330$.

Completing the square,

$$x^2+49x+\left(\frac{49}{2}\right)^2=330+\frac{2401}{4}=\frac{3721}{4}.$$

Extracting the root,

$$x+\frac{49}{2}=\pm\frac{61}{2}\ \therefore x=\frac{-49\pm 61}{2}=6, \text{ or } -55.$$

(83.) $x^4+x^3-4x^2+x+1=0$. This is solved like example 75: thus—

Divide by x^2, then $x^2+x-4+\frac{1}{x}+\frac{1}{x^2}=0$;

that is, $\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)-4=0$

Add 2 to each side, and transpose,

$$\therefore \left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)=6, \text{ or } y^2+y=6$$

$$\therefore y^2+y+\tfrac{1}{4}=\frac{25}{4}\ \therefore y+\tfrac{1}{2}=\pm\frac{5}{2}\ \therefore y=2, \text{ or } -3$$

$$\therefore x+\frac{1}{x}=2, \text{ or } x+\frac{1}{x}=-3$$

$$\therefore x^2-2x=-1, \text{ or } x^2+3x=-1.$$

Completing squares, $x^2-2x+1=0$, or $x^2+3x+\frac{9}{4}=\frac{5}{4}$.

Extracting roots, $x-1=0$, or $x+\frac{3}{2}=\pm\frac{\sqrt{5}}{2}$

$$\therefore x=1, \text{ or } x=\frac{-3\pm\sqrt{5}}{2}.$$

(84.) $2x^2+\sqrt{(x^2+9)}=x^4-9$. Subtract x^2 from each side, then $x^2+\sqrt{(x^2+9)}=x^4-(x^2+9)$.

Dividing each side by the first member, then

$$1=x^2-\sqrt{(x^2+9)}\ \therefore x^2+9=(x^2-1)^2=x^4-2x^2+1$$

$$\therefore x^4-3x^2=8\ \therefore x^4-3x^2+\frac{9}{4}=\frac{41}{4}$$

$$\therefore x^2-\frac{3}{2}=\frac{\sqrt{41}}{2} \therefore x=\sqrt{\frac{3\pm\sqrt{41}}{2}}.$$

These values of x satisfy the equation after the division of it by the factor $x^2+\sqrt{(x^2+9)}$. Equating this to zero, to get the other values, we have

$$x^2+\sqrt{(x^2+9)}=0 \therefore x^4=x^2+9 \therefore x^4-x^2=9$$

$$\therefore x^4-x^2+\tfrac{1}{4}=\frac{37}{4} \therefore x=\sqrt{\frac{1\pm\sqrt{37}}{2}}.$$

(85.) $(a+x)\sqrt{(a^2+x^2)}=6(a-x)^2$

Squaring, $(a+x)^2(a^2+x^2)=36(a-x)^4$;

that is, $a^4+2a^3x+2a^2x^2+2ax^3+x^4=$

$36a^4-144a^3x+216a^2x^2-144ax^3+36x^4$.

Tranposing, $35x^4-146ax^3+214a^2x^2-146a^3x+35a^4=0$;

or $35(x^4+a^4)-146a(x^3+a^2x)+214a^2x^2=0$.

Dividing by x^2,

$$35\left(x^2+\frac{a^4}{x^2}\right)-146a\left(x+\frac{a^2}{x}\right)+214a^2=0.$$

Adding $35\times 2a^2$ to each side, and remembering that

$$x^2+2a^2+\frac{a^4}{x^2}=\left(x+\frac{a^2}{x}\right)^2,$$

$$35\left(x+\frac{a^2}{x}\right)^2-146a\left(x+\frac{a^2}{x}\right)+214a^2=70a^2;$$

$$\text{or } 35y^2-146ay=-144a^2 \therefore y^2-\frac{146a}{35}y=-\frac{144x^2}{35}.$$

Completing the square,

$$y^2-\frac{146a}{35}y+\left(\frac{73a}{35}\right)^2=\frac{5329a^2}{35^2}-\frac{144a^2}{35}=\frac{289a^2}{35^2}.$$

Extracting the root,

$$y-\frac{73a}{35}=\pm\frac{17a}{35} \therefore y=\frac{73a\pm 17a}{35}=\frac{18a}{7}, \text{or } \frac{8a}{5}$$

$$\therefore x+\frac{a^2}{x}=\frac{18a}{7}, \text{or } x+\frac{a^2}{x}=\frac{8a}{5}$$

$$\therefore x^2-\frac{18a}{7}x=-a^2, \text{ or } x^2-\frac{8a}{5}x=-a^2.$$

Completing squares,

$$x^2-\frac{18a}{7}x+\frac{81a^2}{49}=\frac{32a^2}{49}, \text{ or } x^2-\frac{8a}{5}x+\frac{16a^2}{25}=-\frac{9a^2}{25}.$$

Extracting roots,

$$x-\frac{9a}{7}=\pm\frac{\sqrt{(16\times 2)}}{7}a, \text{ or } x-\frac{4a}{5}=\frac{3}{5}a\sqrt{-1}$$

$$\therefore\ x=\frac{9\pm 4\sqrt{2}}{7}a, \text{ or } x=\frac{4\pm 3\sqrt{-1}}{5}a.$$

(86.) $(x+3)^2-2(x^2+3)=2x(x+1)^2$

Transposing, $(x+3)^2=2\{(x^2+3)+x(x+1)^2\}$

$=2\{x^3+3x^2+x+3\}=2\{x^2(x+3)+(x+3)\}.$

Dividing by $x+3$, $x+3=2x^2+2\ \therefore\ 2x^2-x=1$

$$\therefore\ x^2-\frac{1}{2}x=\frac{1}{2}\ \therefore\ x^2-\frac{1}{2}x+\frac{1}{16}=\frac{9}{16}$$

$$\therefore\ x-\frac{1}{4}=\pm\frac{3}{4}\ \therefore\ x=\frac{1\pm 3}{4}=1 \text{ or } -\tfrac{1}{2}.$$

And equating the factor $x+3$ to zero, we have also $x=-3$.

(87.) $\dfrac{x}{x^2+4x}+\dfrac{x}{x^2-3x}=1\tfrac{1}{8}$; that is, $\dfrac{1}{x+4}+\dfrac{1}{x-3}=1\tfrac{1}{8}$.

Clearing fractions, $8(x-3)+8(x+4)=9(x+4)(x-3)$;

that is, $16x+8=9(x^2+x-12)\ \therefore\ 9x^2-7x=116$

$$\therefore\ x^2-\frac{7}{9}x=\frac{116}{9}\ \therefore\ x^2-\frac{7}{9}x+\left(\frac{7}{18}\right)^2=\frac{116}{9}+\frac{49}{18^2}=\frac{4225}{18^2}.$$

Extracting the root,

$$x-\frac{7}{18}=\pm\frac{65}{18}\ \therefore\ x=\frac{7\pm 65}{18}=4, \text{ or } -3\frac{2}{9}.$$

(88.) $x^5-3x^4-9x^3+21x^2-10x+24=0$. Divide by x^2+1, the quotient is $x^3-3x^2-10x+24$, which may be put in the following form: namely,

$$x(x^2-4x+4)+(x^2-14x+14);$$

that is, $x(x-2)^2+(x-2)(x-12)$,

or $(x-2)\{x(x-2)+(x-12)\}$,

or $(x-2)\{x^2-x-12\}$.

Hence the proposed equation is the same as

$$(x^2+1)(x-2)(x^2-x-12)=0,$$

which is satisfied for either

$$x^2+1=0,\ x-2=0, \text{ or } x^2-x-12=0.$$

Taking the last equation, we have

$$x^2-x=12 \therefore x^2-x+\tfrac{1}{4}=\frac{49}{4} \therefore x-\tfrac{1}{2}=\pm\frac{7}{2} \therefore x=4 \text{ or } -3$$

The other two equations give $x=\pm\sqrt{-1}$, and $x=2$.

$$(89.)\ \frac{x^2+1}{x}-\frac{1}{\sqrt{5}}\cdot\frac{x-1}{\sqrt{x}}=4\tfrac{2}{5}.$$

$$\text{or } \left(x+\frac{1}{x}\right)-\frac{1}{\sqrt{5}}\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)=4\tfrac{2}{5}$$

Subtract 2 from each side, then

$$\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2-\frac{1}{\sqrt{5}}\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)=2\tfrac{2}{5}, \text{ or } y^2-\frac{1}{\sqrt{5}}y=\frac{12}{5}.$$

Completing the square,

$$y^2-\frac{1}{\sqrt{5}}y+\frac{1}{20}=\frac{49}{20} \therefore y-\frac{1}{2\sqrt{5}}=\pm\frac{7}{2\sqrt{5}} \therefore y=\frac{4}{\sqrt{5}} \text{ or } \frac{-3}{\sqrt{5}}$$

$$\therefore x-2+\frac{1}{x}=\frac{16}{5} \text{ or } x-2+\frac{1}{x}=\frac{9}{5}$$

$$\therefore x^2-\frac{26}{5}x=-1, \text{ or } x^2-\frac{19}{5}x=-1.$$

Completing squares,

$$x^2-\frac{26}{5}x+\left(\frac{13}{5}\right)^2=\frac{144}{25}, \text{ or } x^2-\frac{19}{5}x+\left(\frac{19}{10}\right)^2=\frac{261}{100}.$$

$$\text{Extracting roots, } x-\frac{13}{5}=\pm\frac{12}{5}, \text{ or } x-\frac{19}{10}=\pm\frac{3\sqrt{29}}{10}$$

$$\therefore x=5, \text{ or } \tfrac{1}{5}, \text{ or } \frac{19\pm3\sqrt{29}}{10}$$

$$(90.)\ 16(x^2+2)^{\frac{3}{2}}+\frac{3}{\sqrt{(x^2+2)}}=32x^2+48.$$

Add 16 to both sides, then

$$16(x^2+2)^{\frac{3}{2}}+\frac{3}{(x^2+2)^{\frac{1}{2}}}+16=32(x^2+2).$$

Put $(x^2+2)^{\frac{1}{2}}=y$, then $16y^3+\dfrac{3}{y}+16=32y^2$

$$\therefore\ 16y^4-32y^3+16y+3=0.\dots(1).$$

Divide by $2y-3$ $\therefore\ 8y^3-4y^2-6y-1=0$.

Divide this by $2y+1$ $\therefore\ 4y^2-4y-1=0$.

Hence the equation (1) is the same as

$$(2y-3)(2y+1)(4y^2-4y-1)=0;$$

which is satisfied for either

$$2y-3=0,\ 2y+1=0,\ \text{or}\ 4y^2-4y-1=0.$$

The two former give $y=\frac{3}{2}$, and $y=-\frac{1}{2}$. From the third,

$$y^2-y=\tfrac{1}{4}\ \therefore\ y^2-y+\tfrac{1}{4}=\tfrac{1}{2}\ \therefore\ y-\tfrac{1}{2}=\frac{\sqrt{2}}{2}\ \therefore\ y=\frac{1\pm\sqrt{2}}{2}.$$

Consequently, $x^2+2=\dfrac{9}{4}$, or $x^2+2=\dfrac{1}{4}$, or $x^2+2=\dfrac{3\pm\sqrt{2}}{4}$

$$\therefore\ x=\sqrt{\frac{1}{4}}=\pm\frac{1}{2},\ \text{or}\ x=\frac{\sqrt{-7}}{2},\ \text{or}\ x=\frac{-5\pm\sqrt{2}}{2}.$$

The first of these is the only value of x given in the book.

(91.) $x^{2m}-a=\dfrac{a}{x^m}+1$. Transposing, $x^{2m}-1=a\left(1+\dfrac{1}{x^m}\right)$

$$\therefore\ x^m(x^{2m}-1)=a(x^m+1)\ \therefore\ x^m(x^m-1)=a;$$

that is, $x^{2m}-x^m=a\ \therefore\ x^{2m}-x^m+\frac{1}{4}=a+\frac{1}{4}$

$$\therefore\ x^m-\tfrac{1}{2}=\sqrt{(a+\tfrac{1}{4})}\ \therefore\ x=\sqrt[m]{\{\tfrac{1}{2}\pm\sqrt{(a+\tfrac{1}{4})}\}}.$$

(92.) $x^4-8x^3+10x^2+24x+5=0$. Dividing by $x-5$, we have

$$x^3-3x^2-5x-1=0;$$

that is, $(x^3+1)-3x(x+1)-2(x+1)=0$.

Dividing this by $x+1$,

$$x^2-x+1-3x-2=0\ \therefore\ x^2-4x-1=0.$$

Consequently the equation is the same as

$$(x-5)(x+1)(x^2-4x-1)=0,$$

which is satisfied for either

$$x-5=0,\ \text{or}\ x+1=0,\ \text{or}\ x^2-4x-1=0.$$

The first two give $x=5$, $x=-1$; and from the third,

$$x^2-4x=1\ \therefore\ x^2-4x+4=5\ \therefore\ x-2=\sqrt{5}\ \therefore\ x=2\pm\sqrt{5}.$$

(93.) $\dfrac{1+x^3}{(1+x)^3}=\frac{1}{3}$; that is, $\dfrac{1-x+x^2}{(1+x)^2}=\frac{1}{3}$

$$\therefore\ 3(1-x+x^2)=1+2x+x^2$$

$$\therefore\ 2x^2-5x=-2\ \therefore\ x^2-\frac{5}{2}x=-1$$

$$\therefore\ x^2-\frac{5}{2}x+\frac{25}{16}=\frac{9}{16}\ \therefore\ x-\frac{5}{4}=\pm\frac{3}{4}\ \therefore\ x=2,\text{ or }\tfrac{1}{2}.$$

(94.) $\dfrac{(x-a)^2}{\sqrt{x}}+2(x-a)=\dfrac{a^2}{\sqrt{x}}+1.$ Multiplying by $\sqrt{x}$,

$(x-a)^2+2(x-a)\sqrt{x}=a^2+\sqrt{x}$. Add x to each side,

$$\therefore\ (x-a)^2+2(x-a)\sqrt{x}+x=a^2+\sqrt{x}+x;$$

that is, $(x-a+\sqrt{x})^2=a^2+x+\sqrt{x}$. Put $x+\sqrt{x}=y$, then

$$(y-a)^2=y+a^2\ \therefore\ y^2-2ay=y\ \therefore\ y-2a=1\ \therefore\ y=2a+1$$

$$\therefore\ x+\sqrt{x}=2a+1\ \therefore\ x+\sqrt{x}+\tfrac{1}{4}=2a+\tfrac{5}{4}$$

$$\therefore\ \sqrt{x}+\tfrac{1}{2}=\sqrt{(2a+\tfrac{5}{4})}\ \therefore\ x=\{-\tfrac{1}{2}\pm\sqrt{(2a+\tfrac{5}{4})}\}^2;$$

that is, $x=2a+\frac{3}{2}\pm\sqrt{(2a+\frac{5}{4})}$.

(95.) $\sqrt{(x^2-1)}+\dfrac{x\sqrt{(x-1)}}{\sqrt{(x+1)}}=\dfrac{\sqrt{(x+1)^3}}{\sqrt{(x-1)}}.$ Multiply the terms of the first fraction by $\sqrt{(x-1)}$, and those of the second by $\sqrt{(x+1)}$, then

$\sqrt{(x^2-1)}+\dfrac{x(x-1)}{\sqrt{(x^2-1)}}=\dfrac{(x+1)^2}{\sqrt{(x^2-1)}}.$ Clearing fractions,

$$x^2-1+x^2-x=x^2+2x+1\ \therefore\ x^2-3x=2\ \therefore\ x^2-3x+\frac{9}{4}=\frac{17}{4}$$

$$\therefore\ x-\frac{3}{2}=\frac{\sqrt{17}}{2}\ \therefore\ x=\frac{3\pm\sqrt{17}}{2}.$$

(96.) $(1+x^3)(1+x^2)(1+x)=30x^3$. By actually performing the multiplications, and transposing, we have the reciprocal equation

$$x^6+x^5+x^4-28x^3+x^2+x+1=0$$

$\therefore\ (x^6+1)+(x^5+x)+(x^4+x^2)=28x^3$. Divide by x^3,

$\therefore\ \left(x^3+\dfrac{1}{x^3}\right)+\left(x^2+\dfrac{1}{x^2}\right)+\left(x+\dfrac{1}{x}\right)=28.$ Add 2,

$$\therefore\ \left(x^3+\frac{1}{x^3}\right)+\left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)=30,$$

$$\text{or}\left(x+\frac{1}{x}\right)\left\{\left(x^2-1+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)+1\right\}=30;$$

that is, $\left(x+\frac{1}{x}\right)\left\{\left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)-2\right\}=30$,

by adding and subtracting 3.

Now it is seen at a glance that this equation is satisfied for

$$x+\frac{1}{x}=3 \therefore x^2-3x=-1 \therefore x^2-3x+\frac{9}{4}=\frac{5}{4}$$

$$\therefore x-\frac{3}{2}=\frac{\sqrt{5}}{2} \therefore x=\frac{3\pm\sqrt{5}}{2}.$$

These are the values of x given in the book. To find the other values, put $x+\frac{1}{x}=y$; then, dividing

$y^3+y^2-2y-30$ by $y-3$, we have for quotient

$$y^2+4y+10=0 \therefore y^2+4y=-10$$

$$\therefore y^2+4y+4=-6 \therefore y+2=\sqrt{-6} \therefore y=-2\pm\sqrt{-6}$$

$$\therefore x+\frac{1}{x}=-2\pm\sqrt{-6} \therefore x^2+(2\mp\sqrt{-6})x=-1$$

$$\therefore x^2+(2\mp\sqrt{-6})x+\frac{(2\mp\sqrt{-6})^2}{4}=\frac{-3\mp2\sqrt{-6}}{2}$$

$$\therefore x+\frac{2\mp\sqrt{-6}}{2}=\sqrt{\frac{-3\mp2\sqrt{-6}}{2}}$$

$$\therefore x=\frac{-2\pm\sqrt{-6}}{2}\pm\sqrt{\frac{-3\mp2\sqrt{-6}}{2}}.$$

(97.) $\frac{x^4}{2}+\frac{17x^3}{4}-17x=8$. Multiplying by 2,

$$x^4+\frac{17x^3}{2}-34x=16 \therefore x^4+\frac{17x^3}{2}=34x+16;$$

that is, $x^4+\frac{17x}{2}x^2=34x+16$. Add $\left(\frac{17x}{4}\right)^2$ to each side, then

$$x^4+\frac{17x}{2}x^2+\left(\frac{17x}{4}\right)^2=\left(\frac{17x}{4}\right)^2+34x+16.$$

Extracting the square root of each side,

$$x^2+\frac{17x}{4}=\pm\left(\frac{17x}{4}+4\right) \therefore x^2=4, \text{ or } x^2+\frac{17x}{2}=-4.$$

From the first of these, we have $x=\pm2$. From the second, we have, by completing the square,

$$x^2+\frac{17x}{2}+\left(\frac{17}{4}\right)^2=\frac{289}{16}-4=\frac{225}{16}.$$

Extracting the root,

$$x+\frac{17}{4}=\pm\frac{15}{4}\ \therefore\ x=\frac{-17\pm15}{4}=-8, \text{ or } -\tfrac{1}{2}.$$

These are the values given in the book. The other two values are $x=\pm2$.

$$(98.)\ x^2(x^2-23)=10x(x^2-24)+649$$
$$\therefore\ x^4-10x^3-23x^2+240x-649=0.$$

Extract the square root of the first member: thus,

$$
\begin{array}{lll}
 & x^4-10x^3-23x^2+240x-649 & (x^2-5x-24 \\
 & x^4 & \\
2x^2-5x) & -10x^3-23x^2 & \\
 & -10x^3+25x^2 & \\
2x^2-10x-24) & \quad -48x^2+240x-649 & \\
 & \quad -48x^2+240x+576 & \\
 & \qquad\qquad -1225=-35^2. &
\end{array}
$$

Consequently, if 35^2 had been added, the first member of the equation would have been a complete square; namely, $(x^2-5x-24)^2$; therefore, the equation is the same as

$$(x^2-5x-24)^2-35^2,$$
$$\text{or } \{(x^2-5x-24)+35\}\{(x^2-5x-24)-35\}=0,$$

which is satisfied for either

$$x^2-5x+11=0, \text{ or } x^2-5x-59=0$$
$$\therefore\ x^2-5x=-11, \text{ or } x^2-5x=59.$$

Completing squares,

$$x^2-5x+\frac{25}{4}=-\frac{19}{4}, \text{ or } x^2-5x+\frac{25}{4}=\frac{261}{4}.$$

Extracting the roots, $x-\dfrac{5}{2}=\dfrac{5\pm\sqrt{-19}}{2}$, or $x-\dfrac{5}{2}=\pm\dfrac{3\sqrt{29}}{2}$

$$\therefore\ x=\frac{10\pm\sqrt{-19}}{2}, \text{ or } x=\frac{5\pm3\sqrt{29}}{2}.$$

The second pair of values are those given in the book.

(99.) $\dfrac{1+x^3}{(1+x)^3}+\dfrac{1-x^3}{(1-x)^3}=a.$ Clearing fractions,

$$(1-x)^3(1+x^3)+(1+x)^3(1-x^3)=a(1-x^2)^3,$$

or $(1+x)^3+(1-x)^3-x^3\{(1+x)^3-(1-x)^3\}=a(1-x^2)^3$;

that is, $2+6x^2-x^3(6x+2x^3)=a(1-x^2)^3$

or $2(1+3x^2-3x^4-x^6)=2\{(1-x^6)+3x^2(1-x^2)\}=a(1-x^2)^3$.

Dividing by $1-x^2$,

$$2\{(1+x^2+x^4)+3x^2\}=2(1+4x^2+x^4)=a(1-x^2)^2;$$

that is, $2(1+4x^2+x^4)=a(1-2x^2+x^4)$

$$\therefore (a-2)x^4-2(a+4)x^2=2-a \;\therefore x^4-\frac{2(a+4)}{a-2}x^2=-1.$$

Completing the square,

$$x^4-\frac{2(a+4)}{a-2}x^2+\left(\frac{a+4}{a-2}\right)^2=\left(\frac{a+4}{a-2}\right)^2-1=\frac{12(a+1)}{(a-2)^2}.$$

Extracting the root,

$$x^2-\frac{a+4}{a-2}=\frac{2\sqrt{\{3(a+1)\}}}{a-2} \quad \therefore x=\left\{\frac{a+4\pm2\sqrt{\{3(a+1)\}}}{a-2}\right\}^{\frac{1}{2}}$$

NOTE.—This example may be treated a little differently by simplifying the fractions at the outset; that is, by expunging the factor $1+x$ from the terms of the first, and the factor $1-x$ from the terms of the second fraction.

(100.) $x+7\sqrt[3]{x}=22$, or putting y for $\sqrt[3]{x}$, $y^3+7y=22$, which equation is obviously satisfied for $y=2$: hence, dividing $y^3+7y-22$ by $y-2$, we have for quotient

$$y^2+2y+11=0 \;\therefore y^2+2y=-11 \;\therefore y^2+2y+1=-10$$

$$\therefore y+1=\sqrt{-10} \therefore y=-1\pm\sqrt{-10} \therefore x=\{-1\pm\sqrt{-10}\}^3.$$

The other value of x, derived from the equation $y-2=0$, is $x=\sqrt[3]{2}$.

(101.) $\sqrt{x}-\dfrac{7}{\sqrt{x}-2}=\dfrac{8}{x}$. Put y for $\sqrt{x}$, then $y-\dfrac{7}{y-2}=\dfrac{8}{y^2}$

Clearing fractions, $y^4-2y^3-7y^2-8y+16=0 \ldots$ (A);

that is, $y^2(y^2-2y+1)-8(y^2+y-2)=0$.

Each of these two terms is obviously divisible by $y-1$; hence, dividing, we have

$$y^2(y-1)-8(y+2)=0, \text{ or } y^3-y^2-8y-16=0;$$

that is, $y(y^2-16)-(y^2-8y+16)=0$.

The first member is evidently divisible by $y-4$, the quotient being $y(y+4)-(y-4)=0$

$$\therefore y^2+3y=-4 \;\therefore y^2+3y+\frac{9}{4}=-\frac{7}{4}$$

$$\therefore y+\frac{3}{2}=\frac{\sqrt{-7}}{2} \;\therefore y=\frac{-3\pm\sqrt{-7}}{2}.$$

Hence all the roots of the proposed equation are

$$x=1,\, x=16,\, x=\left(\frac{-3\pm\sqrt{-7}}{2}\right)^2=\frac{2\mp\sqrt{-7}}{4}.$$

Otherwise.—Extracting the square root of (A), we have

$$\begin{array}{r|l}
 & y^4-2y^3-7y^2-8y+16(y^2-y+4 \\
 & y^4 \\
2y^2-y) & \overline{\quad} -2y^3-7y^2 \\
 & -2y^3+\ y^2 \\
2y^2-2y+4) & \overline{\quad\quad} -8y^2-8y+16 \\
 & \quad 8y^2-8y+16 \\
 & \overline{\quad -16y^2}
\end{array}$$

It thus appears that if $16y^2$ were added to the first member of (A), that member would be a complete square; hence the equation (A) is the same as

$$(y^2-y+4)^2-16y^2=0$$

$$\therefore \{(y^2-y+4)+4y\}\{(y^2-y+4)-4y\}=0$$

$$\therefore y^2+3y+4=0, \text{ or } y^2-5y+4=0$$

$$\therefore y^2+3y+\frac{9}{4}=-\frac{7}{4}, \text{ or } y^2-5y+\frac{25}{4}=\frac{9}{4}$$

$$\therefore y+\frac{3}{2}=\frac{\sqrt{-7}}{4}, \text{ or } y-\frac{5}{2}=\pm\frac{3}{2}$$

$$\therefore y=\frac{-3\pm\sqrt{-7}}{2}, \text{ or } y=\frac{5\pm 3}{2}=1 \text{ or } 4,$$

which are the four values of y determined above.

$$(102.)\ x^4\sqrt{x}+2x^3+35x\sqrt{x}+34=\frac{72}{x\sqrt{x}}.$$

Put $x\sqrt{x}=y$, then $y^2=x^3$, and the equation becomes

$$y^3+2y^2+35y+34=\frac{72}{y}$$

$$\therefore\ y^4+2y^3+35y^2+34y-72=0.$$

Extract the square root,

$$\begin{array}{rl}
 & y^4+2y^3+35y^2+34y-72(y^2+y+17 \\
 & y^4 \\
2y^2+y) & 2y^3+35y^2 \\
 & 2y^3+\quad y^2 \\
2y^2+2y+17) & 34y^2+34y-72 \\
 & 34y^2+34y+289 \\
 & \qquad\qquad -361=-19^2.
\end{array}$$

Hence, if 19^2 be added to the first member of the equation, it will be a cemplete aquare;

$$\therefore\ (y^2+y+17)^2-19^2=0;\ \text{that is,}$$

$$\{(y^2+y+17)+19\}\{y^2+y+17)-19\}=0$$

$$\therefore\ y^2+y+36=0,\ \text{or}\ y^2+y-2=0$$

$$\therefore\ y^2+y+\tfrac{1}{4}=-\frac{143}{4},\ \text{or}\ y^2+y+\tfrac{1}{4}=\frac{9}{4}$$

$$\therefore\ y+\tfrac{1}{2}=\frac{\sqrt{-143}}{2},\ \text{or}\ y+\tfrac{1}{2}=\pm\frac{3}{2}$$

$$\therefore\ y=\frac{-1\pm\sqrt{-143}}{2},\ \text{or}\ y=\frac{-1\pm3}{2}=1\ \text{or}\ -2$$

$$\therefore\ x=y^{\frac{2}{3}}=\left\{\frac{-1\pm\sqrt{-143}}{2}\right\}^{\frac{2}{3}},\ \text{or}\ x=1,\ \text{or}\ \sqrt[3]{-4}.$$

NOTE.—This method of solution, which is the same as that employed in examples 98 and 101, deserves attention, as well on account of its novelty, as because of the ease and expedition with which it enables us, in many instances, to decompose an equation into two others, each of half the degree of the original. The object aimed at is to give to the partial square root, or what may be called the imperfect square root, such a form, that the remainder at which the operation stops may be either a complete square or a mere number. In example 101, this is effected by putting 4 for the final term of the imperfect root instead of -4, as the hackneyed method of extracting the square root would direct: but it is plain that what is here called the imperfect root admits of an endless variety of forms,

as well as of the form usually fixed on, each form giving rise to its own peculiar remainder. Whenever a form can be discovered that will render this corresponding remainder a square, or a number, the decomposition of the equation may be effected, as in the examples referred to, the factors of it being the imperfect root $\pm$ the square root of the remainder after changing its sign.

Quadratic Equations, with two or more Unknown Quantities (Page 88).

(1.) $\left.\begin{array}{l} x^2+y^2=20. \\ x^2+y^2=12. \end{array}\right\}$ By adding and subtracting, $2x^2=32$ and $2y^2=8$

$\therefore x=\pm 4,\ y=\pm 2.$

(2.) $\left.\begin{array}{l} x+y=6. \\ x^2+y^2=26. \end{array}\right\}$ Squaring the first, $x^2+2xy+y^2=36$

Subtracting the second, $x^2 \quad +y^2=26$

$2xy \quad =10$

Subtracting this from the second, $x^2-2xy+y^2=16$

$\therefore x-y=\pm 4$; also, $x+y=6 \therefore x=\dfrac{6\pm 4}{2},\ y=\dfrac{6\pm 4}{2}$;

that is, $x=5$ or 1; $y=1$, or 5.

(3.) $\left.\begin{array}{l} x^2+y^2=10. \\ x-y=2. \end{array}\right\}$ Squaring the second, $x^2-2xy+y^2=4$

Subtracting this from $x^2 \quad +y^2=10$

$2xy \quad =6$

Adding to the preceding, $x^2+2xy+y^2=16$

$\therefore x+y=\pm 4$; also, $x-y=2 \therefore x=\dfrac{2\pm 4}{2},\ y=\dfrac{-2\pm 4}{2}$;

that is, $x=3$ or -1; $y=1$, or -3.

(4.) $\left.\begin{array}{l} x^2+y^2=25. \\ x+y=1. \end{array}\right\}$ Squaring the second, $x^2+2xy+y^2=\ 1$

Subtracting the first, $x^2 \quad +y^2=\ 25$

$2xy \quad =-24$

Subtracting this from the preceding, $x^2-2xy+y^2=\ 49$

$\therefore x-y=\pm 7$; also, $x+y=1 \therefore x=\dfrac{1\pm 7}{2},\ y=\dfrac{1\mp 7}{2}$;

that is, $x=4$, or -3; $y=-3$, or 4.

(5.) $\left.\begin{array}{l} x^2-y^2=16. \\ x+y=8. \end{array}\right\}$ Dividing the first by the second, $x-y=2$, also $x+y=8$

$$\therefore x=\frac{8+2}{2},\ y=\frac{8-2}{2} \ \therefore x=5, y=3.$$

(6.) $\left.\begin{array}{l} x-y=1. \\ x^3-y^3=19. \end{array}\right\}$ Cubing the first, $x^3-y^3-3xy(x-y)$ or $x^3-y^3-3xy=1$.

Subtracting this from the second, $3xy=18$

$\therefore 4xy=24$, also from the first, $x^2-2xy+y^2=1$

Adding $\qquad 4xy \quad =24$

$x^2+2xy+y^2=25.$

$\therefore x+y=\pm 5$, also $x-y=1$
$\therefore x=3$ or -2, $y=2$, or 3.

(7.) $\left.\begin{array}{l} x^3+y^3=189, \\ x^2y+xy^2=180. \end{array}\right\}$ Add 3 times the second to the first, then

$$x^3+y^3+3xy(x+y)=729;$$

that is, $(x+y)^3=369 \ \therefore x+y=9 \quad \therefore (x+y)^2=81$

From the second equation,

$(x+y)xy=180 \ \therefore 9xy=180 \ \therefore \quad 4xy=80$

$\therefore (x-y)^2=1$

$\therefore x-y=\pm 1$, also $x+y=9$
$\therefore x=5$, or 4; $y=4$, or 5.

(8.) $\left.\begin{array}{l} 10x+y=3xy. \\ y-x=2. \end{array}\right\}$ From the second, $y=x+2$. Substituting in the first,

$$11x+2=3x^2+6x \ \therefore 3x^2-5x=2$$

$$\therefore x^2-\frac{5}{3}x+\frac{25}{36}=\frac{2}{3}+\frac{25}{36}=\frac{49}{36}$$

$$\therefore x-\frac{5}{6}=\pm\frac{7}{6} \ \therefore x=\frac{5\pm 7}{6}=2, \text{ or } -\frac{1}{3}$$

$$\therefore y=x+2=4, \text{ or } \frac{5}{3}.$$

(9.) $\left.\begin{array}{r} x^2+y^2+x+y=18. \\ 2xy=12. \end{array}\right\}$ Adding, $(x+y)^2+(x+y)=30$

$$\therefore (x+y)^2+(x+y)+\tfrac{1}{4}=\frac{121}{4} \ \therefore x+y+\tfrac{1}{2}=\pm\frac{11}{2}$$

$$\therefore x+y=5, \text{ or } -6 \therefore (x+y)^2 = 25, \text{ or } 36$$
$$\text{also } 4xy = 24 \quad 24$$
$$\therefore (x-y)^2 = 1 \text{ or } 12$$
$$\therefore (x-y = \pm 1 \text{ or } \pm 2\sqrt{3}$$
$$\text{also } x+y = 5 \text{ or } -6$$
$$\therefore \begin{cases} x = 3 \text{ or } 2, \text{ or } -3 \pm \sqrt{2} \\ y = 2 \text{ or } 3, \text{ or } -3 \mp \sqrt{3} \end{cases}$$

(10.) $\left.\begin{array}{l} x^2y^4+y^2=10. \\ xy^4+y=4. \end{array}\right\}$ From the first equation,

$$x^2y^4=10-y^2$$

From the second, $x^2y^4=\dfrac{(4-y)^2}{y^4}$

$$\therefore 10-y^2=\frac{16-8y+y^2}{y^4} \therefore y^6-10y^4+y^2-8y+16=0.$$

This may be put in the following form: namely,

$$y^4(y^2-1)-9y^2(y^2-1)-8(y^2+y-2)=0.$$

Each term of the first member is obviously divisible by $y-1$; hence the equation is satisfied for

$$y-1=0 \therefore y=1 \therefore x=\frac{4-y}{y^4}=3.$$

(11.) $\left.\begin{array}{l} \dfrac{x+y}{x-y}=a^2. \\ x^2-y^2=b^2. \end{array}\right\}$ Multiplying the two together,

$$(x+y)^2=a^2b^2$$

$$\therefore x+y=ab \therefore \frac{x^2-y^2}{x+y}=\frac{b^2}{ab} \therefore x-y=\frac{b}{a}$$

$$\therefore x=\tfrac{1}{2}\left(ab+\frac{b}{a}\right), \ y=\tfrac{1}{2}\left(ab-\frac{b}{a}\right);$$

that is, $x=\dfrac{b}{2a}(a^2+1)$, $y=\dfrac{b}{2a}(a^2-1)$.

(12.) $\left.\begin{array}{l} 9x^2=4y^2. \\ 3xy+2x+y=485. \end{array}\right\}$ From the first, $3x=2y$. Substituting in the second, we have $2y^2+\frac{4}{3}y+y=485 \therefore y^2+\frac{7}{6}y=\frac{485}{2}$

$$\therefore y^2+\frac{7}{6}y+\left(\frac{7}{12}\right)^2=\frac{34969}{12^2} \therefore y+\frac{7}{12}=\pm\frac{187}{12}$$

$$\therefore y=\frac{-7\pm 187}{12}=15,\text{or}-16\tfrac{1}{6}\ \therefore x=\frac{2y}{3}=10,\text{ or }-10\frac{7}{9}.$$

(13.) $\left.\begin{aligned}x^2+y^2-x-y&=78.\\ xy+x+y&=39.\end{aligned}\right\}$ Add the first to twice the second, and then subtract;

$$\therefore x^2+2xy+y^2+x+y=156 \quad\ldots.\ (1)$$

$$\text{and } x^2-2xy+y^2-3(x+y)=0\ldots.\ (2)$$

$$\therefore (x+y)^2+(x+y)+\tfrac{1}{4}=\frac{625}{4}\ \therefore x+y+\tfrac{1}{2}=\pm\frac{25}{2}$$

$$\therefore x+y=12,\text{ or }-13.$$

Hence, substituting in (2)

$$(x-y)^2=36,\text{ or }-39\ \therefore x-y=\pm\ 6\text{ or }\pm\sqrt{-39}$$
$$\text{also}\quad x+y=\ 12\text{ or }-13$$

$$\therefore x=9\quad\text{or}\quad 3,\text{ or }\frac{-13\pm\sqrt{-39}}{2}$$

$$y=3\quad\text{or}\quad 9,\text{ or }\frac{-13\mp\sqrt{-39}}{2}$$

(14.) $\left.\begin{aligned}\frac{1}{y}-\frac{1}{x}&=\frac{1}{4}.\\ x^2y-xy^2&=16.\end{aligned}\right\}$ Multiplying the two together, $x^2-2xy+y^2=4\ \therefore\ x-y=2,$ also $xy(x-y)=16$

$\therefore xy=8.$ From $(x-y)^2=\ 4$
add $\quad 4xy\ =32$

$$\therefore (x+y)^2=36\ \therefore x+y=\pm\ 6$$
$$\text{also}\quad x-y=\quad 2$$

$$\therefore\quad x\ =\ 4\text{ or }-2$$
$$y\ =\ 2\text{ or }-4.$$

(15.) $\left.\begin{aligned}x^2+xy&=a^2+ab.\\ y^2+yx&=b^2+ab.\end{aligned}\right\}$ Adding the two together, we have

$$x^2+2xy+y^2=a^2+2ab+b^2\ \therefore x+y=a+b.$$

Also, $x(x+y)=a^2+ab\ \therefore (a+b)x=a(a+b)\ \therefore x=a\ \therefore y-b.$

(16.) $\left.\begin{aligned}12xy&=5x+12y.\\ y^2-x^2&=1.\end{aligned}\right\}$ From the first, $12(x-1)y=5x.$ From the second, $y=\sqrt{(x^2+1)}.$

$$\text{Substituting, } 12(x-1)\sqrt{(x^2+1)}=5x$$

$$\therefore 144(x-1)^2(x^2+1)=25x^2$$

$$\therefore 144(x^4-2x^3+2x^2-2x+1)=25x^2;$$

$$\text{that is, } 144(x^4+1)-288(x^3+x)=-263x^2$$

$$\therefore 144\left(x^2+\frac{1}{x^2}\right)-288\left(x+\frac{1}{x}\right)=-263.$$

$$\text{Adding } 144\times 2,\ 144\left(x+\frac{1}{x}\right)^2-288\left(x+\frac{1}{x}\right)=25\ ;$$

$$\text{or, } 144y^2-288y=25 \ \therefore y^2-2y=\frac{25}{144}$$

$$\therefore y^2-2y+1=\frac{169}{144} \ \therefore y-1=\pm\frac{13}{12} \ \therefore y=\frac{25}{12}, \text{ or } -\frac{1}{12}$$

$$\therefore x+\frac{1}{x}=\frac{25}{12}, \text{ or } x+\frac{1}{x}=-\frac{1}{12}$$

$$\therefore x^2-\frac{25}{12}x=-1, \text{ or } x^2+\frac{1}{12}x=-1$$

$$\therefore x^2-\frac{25}{12}x+\left(\frac{25}{24}\right)^2=\frac{49}{24^2}, \text{ or } x^2+\frac{1}{12}x+\frac{1}{24^2}=-\frac{552}{24^2}$$

$$\therefore x-\frac{25}{24}=\pm\frac{7}{25}, \text{ or } x+\frac{1}{24}=\pm\frac{\sqrt{-552}}{24}$$

$$\therefore x=\frac{25\pm 7}{24} \text{ or } x=\frac{-1\pm\sqrt{-552}}{24};$$

$$\text{that is, } x=1\tfrac{1}{3} \text{ or } \tfrac{3}{4}\ ; \text{ or } x=\frac{-1\pm\sqrt{-552}}{24}=\frac{-1\pm 2\sqrt{-138}}{24}$$

$$\therefore y=\sqrt{(x^2+1)}=1\tfrac{2}{3}, \text{ or } 1\tfrac{1}{4}\ ; \text{ or } y=\frac{\sqrt{(25\pm 2\sqrt{-138})}}{24}.$$

(17.) $\left.\begin{array}{l} 2y+3x=8. \\ 3y^2+2x^2=11. \end{array}\right\}$ From the first, $y=\dfrac{8-3x}{2}$.

Substituting this in the second, $3\left(\dfrac{8-3x}{2}\right)^2+2x^2=11$:

$$\text{that is, } 35x^2-144x=-148$$

$$\therefore x^2-\frac{144}{35}x=-\frac{148}{35}$$

$$\therefore x^2-\frac{144}{35}x+\left(\frac{72}{35}\right)^2=\frac{4}{35^2} \ \therefore x-\frac{72}{35}=\pm\frac{2}{35} \ \therefore x=2, \text{ or } 2\frac{4}{35}$$

$$\therefore y=4-\frac{3x}{2}=1, \text{ or } \frac{29}{35}.$$

(18.) $\left.\begin{array}{l} y-x=2. \\ 3xy=10x+y. \end{array}\right\}$ From the first, $y=x+2$. Substituting this in the second,

$$3x^2+6x=11x+2$$

$$\therefore x^2-\frac{5}{3}x=\frac{2}{3} \therefore x^2-\frac{5}{3}x+\frac{25}{36}=\frac{49}{36}$$

$$\therefore x-\frac{5}{6}=\pm\frac{7}{6} \therefore x=2, \text{ or } -\frac{1}{3}.$$

(19.) $\left.\begin{array}{l} x+y+\sqrt{(x+y)}=12. \\ x^3+y^3=189. \end{array}\right\}$ From the first, completing the square,

$$(x+y)+(x+y)^{\frac{1}{2}}+\tfrac{1}{4}=\frac{49}{4} \therefore (x+y)^{\frac{1}{2}}=\frac{-1\pm 7}{2}=3 \text{ or } -4.$$

The second is $(x+y)(x^2-xy+y^2)=189$; that is, it is either

$$9(x^2-xy+y^2)=189, \text{ or } 16(x^2-xy+y^2)=189\ldots.\text{ (A)}.$$

Substituting $9-x$ for y in the former of these, after dividing by 9, we have $3x^2-27x=-60 \therefore x^2-9x=-20$

$$\therefore x^2-9x+\frac{81}{4}=\tfrac{1}{4} \therefore x-\frac{9}{2}=\pm\tfrac{1}{2} \therefore x=5, \text{ or } 4$$

$$\therefore y=9-x=4, \text{ or } 5.$$

These are the values in the book; another pair may be obtained from the second of the equations (A).

(20.) $\left.\begin{array}{l} 4xy=96-x^2y^2. \\ x+y=6. \end{array}\right\}$ From the first, $(xy)^2+4xy=96$.

Completing the square, $(xy)^2+4xy+4=100$

$$\therefore xy+2=\pm 10 \therefore xy=8, \text{ or } -12.$$

From the second equation, $(x+y)^2=36$

also, $4xy=32$ or -48

$$\therefore (x-y)^2=4 \text{ or } 84$$

$$\therefore x-y=\pm 2, \text{ or } 2\sqrt{21}; \text{ also } x+y=6$$

$$\therefore x=4, \text{ or } 2; \text{ or } 3\pm\sqrt{21}$$

$$y=2, \text{ or } 4; \text{ or } 3\mp\sqrt{21}.$$

(21.) $\left.\begin{array}{l} \dfrac{x^2+y^2}{xy}=\dfrac{13}{6}. \\ x^3y+xy^3=x^2y^2+42. \end{array}\right\}$ From the first, $x^2+y^2=\frac{13}{6}xy\ldots.$ (A)

From the second, $xy(x^2+y^2-xy)=42$.

By substitution,

$$xy\left(\frac{13}{6}xy-xy\right)=42\text{; that is, }\frac{7}{6}(xy)^2=42\ \therefore\ xy=6$$

$$\therefore \text{ from (A), } x^2+y^2=13\ \therefore\ x^2+2xy+y^2=25\ \therefore\ x+y=\pm 5$$

$$\text{and }\ x^2-2xy+y^2=\ 1\ \therefore\ x-y=\pm 1$$

$$\therefore\ x=3 \text{ or } -3,\ y=2, \text{ or } -2.$$

(22.) $\left.\begin{array}{l} x^2+y^2+4x-6y=13. \\ xy-3x+2y=11. \end{array}\right\}$ Add twice the second to the first, then

$$x^2+2xy+y^2-2(x+y)=35.$$

Completing the square, $(x+y)^2-2(x+y)+1=36$

$$\therefore\ x+y-1=\pm 6\ \therefore\ x+y=7, \text{ or } -5$$

$$\therefore\ y=7-x, \text{ or } y=-5-x.$$

Substituting in the second equation,

$$7x-x^2-3x+14-2x=11, \text{ or } -5x-x^2-3x-10-2x=11;$$

$$\text{that is, } -x^2+2x=-3, \text{ or } -x^2-10x=21$$

$$\therefore\ x^2-2x+1=4, \text{ or } x^2+10x+25=4$$

$$\therefore\ x-1=\pm 2, \text{ or } x+5=\pm 2\ \therefore\ x=3, \text{ or } -1; \text{ or } -3, \text{ or } -7$$

$$\therefore\ y=7-x=4, \text{ or } 8;\ y=-5-x=-2, \text{ or } 2.$$

(23.) $\left.\begin{array}{r} (x^2+y^2)x^2y^2=3600 \\ x^2y+xy^2=84 \end{array}\right\}\ \therefore\ \left\{\begin{array}{r} x^4y^2+x^2y^4 \qquad =3600 \\ x^4y^2+x^2y^4+2x^3y^3=7056 \end{array}\right.$

$$2x^3y^3=3456$$

$$\therefore\ xy=\sqrt[3]{1728}=12\ \therefore \text{ from the first equation, } x^2+y^2=25$$

$$\text{Adding and subtracting,}\qquad 2xy=24$$

$$\therefore\ x+y=\pm 7,\ x-y=\pm 1 \qquad (x+y)^2=49$$

$$\therefore\ x=3, \text{ or } 4;\ y=4, \text{ or } 3 \qquad (x-y)^2=\ 1$$

(24.) $\left.\begin{array}{l} x^4+y^4=2657. \\ x\ +y\ =11. \end{array}\right\}$ This example may be treated exactly like ex. 6, at p. 83 of the Algebra.

(25.) $\left.\begin{array}{r} 2x+2y-x^2-y^2+2=0 \\ xy=3 \end{array}\right\}\ \therefore\ \left\{\begin{array}{r} 2x+2y+2=x^2+y^2 \\ 6=\ 2xy \end{array}\right.$

$$\text{Adding,}\qquad 2(x+y)+8=(x+y)^2$$

$$\therefore\ (x+y)^2-2(x+y)=8\ \therefore\ (x+y)^2-2(x+y)+1=9$$

$$\therefore\ x+y-1=\pm 3\ \therefore\ x+y=4 \text{ or } -2$$

$$\therefore (x+y)^2=16, \text{ or } \quad 4$$
$$4xy \;\; =12 \qquad 12$$

$$(x-y)^2= \;4 \text{ or } -\;8 \therefore x-y=\pm 2, \text{ or } 2\sqrt{-2}.$$

And since $x+y=4$ or -2 $\therefore x=3$ or 1; or $-(1\pm\sqrt{-2})$
and $y=1$ or 3; or $-(1\mp\sqrt{-2})$.

(26.) $\left.\begin{array}{l} x^2+y^2=61. \\ x^2-xy=6. \end{array}\right\}$ This example is similar to that at p. 86 of the Algebra, and may be solved in the same way.

(27.) $\left.\begin{array}{l} x+y=10. \\ x^5+y^5=17050. \end{array}\right\}$ Dividing the second by the first, we have

$$x^4-x^3y+x^2y^2-xy^3+y^4 \;=\; 1705$$
$$(x+y)^4=x^4+4x^3y+6x^2y^2+4xy^3+y^4 \;=10000$$

Subtracting, $\quad 5x^3y+5x^2y^2+5xy^3 \;=\; 8295$

$$\therefore \quad x^2 + xy + y^2 \;=\; \frac{1659}{xy}$$
$$(x+y)^2=x^2+2xy+y^2 \;=\; 100$$
$$\therefore \quad xy \;=\; 100-\frac{1659}{xy}$$

$$\therefore (xy)^2-100xy=-1659 \therefore (xy)^2-100xy+2500=841$$
$$\therefore xy-50=\pm 29 \therefore xy=21, \text{ or } 79$$

$$(x+y)^2=100$$
$$4xy \;=\; 84 \text{ or } 316$$

$(x-y)^2= \;16$ or -216 $\therefore x-y=\pm 4$ or $6\sqrt{-6}$; and since
$x+y \;=\; 10$ $\therefore x=7$, or 3; or $5\pm 3\sqrt{-6}$
$y=3$, or 7; or $5\mp 3\sqrt{-6}$.

(28.) $\left.\begin{array}{l} x^2+xy+y^2=21 \\ x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y=3 \end{array}\right\} \therefore \left\{\begin{array}{l} (x+y)^2-xy=21. \\ (x+y)-(xy)^{\frac{1}{2}}=3. \end{array}\right.$

Divide the first of these by the second, then

$$(x+y)+(xy)^{\frac{1}{2}}=7.$$

Add this to the equation above, then $2(x+y)=10 \therefore x+y=5$.

Also $xy=(x+y)^2-21=4 \therefore (x+y)^2-4xy=(x-y)^2=9$
$\therefore x-y=\pm 3 \therefore x=1$ or 4; $y=4$ or 1.

(29.) $\left.\begin{array}{l} x^2y^2+12xy=9x^2+4y^2 \\ x^2+4x+y^2=6y+24 \end{array}\right\} \therefore \left\{\begin{array}{l} x^2y^2=(3x-2y)^2 \\ x^2+y^2=6y-4x+24 \end{array}\right.$

$$\begin{aligned} \text{To } x^2+y^2&=6y-4x+24 \ldots\ldots (A)\\ \text{add } 2xy &=6x-4y \quad \ldots\ldots (B)\end{aligned}$$

$$\therefore (x+y)^2=2(x+y)+24 \therefore (x+y)^2-2(x+y)=24$$

$$\therefore (x+y)^2-2(x+y)+1=25 \therefore x+y-1=\pm 5$$

$$\therefore x+y=6, \text{ or } -4 \therefore y=6-x, \text{ or } y=-4-x.$$

Substituting in (B), $12x-2x^2=6x-24+4x$, or $-8x-2x^2=6x+16+4x \therefore 2x^2-2x=24$, or $2x^2+18x=-16$

$$\therefore x^2-x+\tfrac{1}{4}=\frac{49}{4}, \text{ or } x^2+9x+\left(\frac{9}{2}\right)^2=\frac{49}{4}$$

$$\therefore x-\tfrac{1}{2}=\pm\frac{7}{2}, \text{ or } x+\frac{9}{2}=\pm\frac{7}{2}$$

$$\therefore x=4, \text{ or } -3; \text{ or } x=-1, \text{ or } -8$$

$$\therefore y=6-x=2, \text{ or } 9; \text{ or } y=-4-x=-3, \text{ or } 4.$$

(30.) $\left.\begin{aligned} x-y&=2.\\ x^4+y^4&=272.\end{aligned}\right\}$ From the first, $x^4-4x^3y+6x^2y^2-4xy^3+y^4=16.$

Subtracting this from the second,

$$\begin{aligned} 4x^3y-6x^2y^2+4xy^3 &= 256\\ \therefore 4x^2-6xy+4y^2 &= \frac{256}{xy}\\ 4(x-y)^2=4x^2-8xy+4y^2 &= 16\\ \therefore 2xy= \frac{256}{xy}-16 \end{aligned}$$

$$\therefore (xy)^2+8xy=128 \therefore (xy)^2+8xy+16=144$$

$$\therefore xy+4=\pm 12 \therefore xy=8 \text{ or } -16.$$

$$\left.\begin{aligned} \text{To } (x-y)^2&=4\\ \text{add } 4xy &=32 \text{ or } =64\end{aligned}\right\} \begin{aligned}&\therefore (x+y)^2=36, \text{ or } -60\\ &\therefore x+y=\pm 6, \text{ or } 2\sqrt{-15}\end{aligned}$$

$$\begin{aligned}\therefore x&=4, \text{ or } -2; \text{ or } 1\pm\sqrt{-15}\\ y&=2, \text{ or } -4; \text{ or } -1\pm\sqrt{-15}.\end{aligned}$$

(31.) $\left.\begin{aligned} x^2+2xy+y+3x&=73.\\ y^2+x+3y&=44.\end{aligned}\right\}$ Adding these together, we have

$$x^2+2xy+y^2+4x+4y=117;$$

$$\text{that is, } (x+y)^2+4(x+y)=117$$

$$\therefore (x+y)^2+4(x+y)+4=121 \therefore x+y+2=\pm 11$$

$$\therefore x+y=9 \text{ or } -13 \therefore x=9-y, \text{ or } -13-y.$$

Hence, by substitution in the second equation,

$$y^2+2y+9=44 \text{ or } y^2+2y-13=44$$
$$\therefore y^2+2y+1=36, \text{ or } y^2+2y+1=58$$
$$\therefore y+1=\pm 6, \text{ or } y+1=\pm\sqrt{58}$$
$$\therefore y=5, \text{ or } -7\,; \text{ or } y=-1\pm\sqrt{58}$$
$$\therefore x=9-y=4, \text{ or } 16\,; \text{ or } x=-13-y=-12\mp\sqrt{58}.$$

(32.) $\left.\begin{array}{l}(x^2+y^2)(x+y)=120.\\ (x-y)(x^2-y^2)=24.\end{array}\right\}$ Dividing the first by the second,

$$\frac{x^2+y^2}{(x-y)^2}=5. \quad \text{Subtract 1, } \therefore \frac{2xy}{(x-y)^2}=4$$
$$\therefore \frac{4xy}{(x-y)^2}=8. \quad \text{Add 1, } \therefore \left(\frac{x+y}{x-y}\right)^2=9 \therefore \frac{x+y}{x-y}=\pm 3$$
$$\therefore x=2y, \text{ or } y=2x.$$

Substituting in the second equation, we have

$$3y^3=24 \therefore y^3=8 \therefore y=2 \therefore x=4$$
$$\text{or } 3x^3=24 \therefore x^3=8 \therefore x=2 \therefore y=4.$$

(33.) $\left.\begin{array}{l}xy=6.\\ 3x^2-7y^2+1=0.\end{array}\right\}$ Assume $x=vy$, then the equations are

$$vy^2=6,\ 3v^2y^2-7y^2=-1$$
$$\therefore y^2=\frac{6}{v},\ y^2=\frac{1}{7-3v^2}$$
$$\therefore 42-18v^2=v \therefore 18v^2+v=42 \therefore v^2+\frac{1}{18}v=\frac{7}{3}$$
$$\therefore v^2+\frac{1}{18}v+\left(\frac{1}{36}\right)^2=\frac{3025}{36^2} \therefore v+\frac{1}{36}=\pm\frac{55}{36} \therefore v=\frac{3}{2}, \text{ or } -\frac{14}{9}$$
$$\therefore y^2=\frac{6}{v}=4 \therefore y=\pm 2 \therefore x=vy=\pm 3.$$

(34.) $\left.\begin{array}{l}2x^{-\frac{1}{4}}y^{-} =1+x^{\frac{3}{4}}y^{\frac{1}{4}}.\\ x-y=\sqrt{(a-x+y)}-\sqrt{(a+x-y)}.\end{array}\right\}$ Squaring the second equation we have

$$(x-y)^2=2a-2\sqrt{\{a^2-(x-y)^2\}}$$
$$\therefore 4\{a^2-(x-y)^2\}=\{2a-(x-y)^2\}^2=$$
$$4a^2-4a(x-y)^2+(x-y)^4$$
$$\therefore 4(x-y)^2=4a(x-y)^2-(x-y)^4$$
$$\therefore \{4(1-a)+(x-y)^2\}(x-y)^2=0.$$

This is satisfied for $x-y=0 \therefore x=y \therefore$ the first equation is $2x^{-\frac{1}{2}}=1+x$, which is evidently satisfied for $x=1$.

(35.) $\left.\begin{array}{l} x-y=3. \\ x^3+y^3=19(x+y). \end{array}\right\}$ Divide the second by $x+y$, then

$x^2-xy+y^2=19$. Subtract the square of the first,

$x^2-2xy+y^2=9$

$\therefore xy=10$. Add 4 times this to the above,

$x^2+2xy+y^2=49 \therefore x+y=\pm 7$

$\therefore x=5$, or -2; $y=2$, or -5.

(36.) $\left.\begin{array}{l} x-2\sqrt{(xy)}+y-\sqrt{x}+\sqrt{y}=0. \\ \sqrt{x}+\sqrt{y}=5. \end{array}\right\}$ The first of these is

$$(\sqrt{x}-\sqrt{y})^2-(\sqrt{x}-\sqrt{y})=0$$

$$\therefore (\sqrt{x}-\sqrt{y})^2-(\sqrt{x}-\sqrt{y})+\tfrac{1}{4}=\tfrac{1}{4}$$

$$\therefore \sqrt{x}-\sqrt{y}-\tfrac{1}{2}=\pm\tfrac{1}{2} \therefore \sqrt{x}-\sqrt{y}=1 \text{ or } 0.$$

Also from the second, $\sqrt{x}+\sqrt{y}=5$

$$\therefore \sqrt{x}=3 \text{ or } \frac{5}{2} \therefore x=9, \text{ or } \frac{25}{4}$$

$$\therefore \sqrt{y}=2 \text{ or } \frac{5}{2} \therefore y=4, \text{ or } \frac{25}{4}.$$

(37.) $\left.\begin{array}{l} \dfrac{x^2}{y^2}+\dfrac{4x}{y}=9\dfrac{4}{9}. \\ x-y=2. \end{array}\right\}$ Multiplying the first by y^2, then

$$x^2+4xy=\frac{85}{9}y^2$$

$$\therefore x^2+4xy+4y^2=\frac{121}{9}y^2$$

$$\therefore x+2y=\pm\frac{11}{3}y \therefore x=\frac{5}{3}y, \text{ or } -\frac{17}{3}y.$$

But from the second, $x=y+2 \therefore y+2=\frac{5}{3}y$, or $y+2=-\frac{17}{3}y$

$$\therefore \tfrac{2}{3}y=2, \text{ or } \frac{20}{3}y=-2 \therefore y=3, \text{ or } -\frac{3}{10}$$

$$\therefore x=y+2=5, \text{ or } \frac{17}{10}.$$

(38.) $\left.\begin{array}{l} x^4-2x^2y+y^2=49 \\ x^4-2x^2y^2+y^4-x^2+y^2=20 \end{array}\right\} \therefore \left\{\begin{array}{l} x^2-y=\pm 7. \dots (A) \\ (x^2-y^2)^2-(x^2-y^2)=20 \end{array}\right.$

$$\therefore (x^2-y^2)^2-(x^2-y^2)+\tfrac{1}{4}=\frac{81}{4} \therefore x^2-y^2-\tfrac{1}{2}=\pm\frac{9}{2}$$

$$\therefore x^2=y^2+\frac{1\pm 9}{2};$$

that is, $x^2=y^2+5$, or y^2-4. Substituting this in (A),

$$y^2-y=2 \text{ or } -12;\ \text{or } y^2-y=11 \text{ or } -3$$

$$\therefore y^2-y+\tfrac{1}{4}=\frac{9}{4}, \text{ or } y^2-y+\tfrac{1}{4}=-\frac{47}{4};$$

$$\text{or } y^2-y+\tfrac{1}{4}=\frac{45}{4}, \text{ or } y^2-y+\tfrac{1}{4}=-\frac{11}{4}$$

$$\therefore y-\tfrac{1}{2}=\pm\tfrac{3}{2}, \text{ or } y-\tfrac{1}{2}=\frac{\sqrt{-47}}{2};$$

$$\text{or } y-\tfrac{1}{2}=\frac{3\sqrt{5}}{2}, \text{ or } y-\tfrac{1}{2}=\frac{\sqrt{-11}}{2}$$

$$\therefore y=2, \text{ or } -1, \text{ or } \frac{1\pm\sqrt{-47}}{2};\ \text{or } y=\frac{1\pm3\sqrt{5}}{2},$$

$$\text{or } \frac{1\pm\sqrt{-11}}{2}\ \therefore x=\sqrt{(y^2+5)}=\pm3, \text{ or } \pm\sqrt{6}, \text{ \&c.}$$

(39.) $\left.\begin{aligned} x^n+y^n&=a.\\ xy&=b.\end{aligned}\right\}$ Squaring the first, $x^{2n}+2x^ny^n+y^{2n}=a^2$

Raising the second, $4x^ny^n=4b^n$

$$\therefore x^n-y^n=a^2-4b^n$$

Combining this with the first by addition, and then taking the nth root, we have

$$x=\sqrt[n]{\tfrac{1}{2}\{\sqrt{(a^2-4b^n)}+a\}}$$

$$\therefore y=\frac{b}{\sqrt[n]{\tfrac{1}{2}\{\sqrt{(a^2-4b^n)}+a\}}}.$$

(40.) $\left.\begin{aligned} &\frac{x^2}{y^2}+\frac{y^2}{x^2}-\frac{3}{2}\left(\frac{x}{y}+\frac{y}{x}\right)=2\frac{3}{50}.\\ &4x-5y=10.\end{aligned}\right\}$ Add 2 to each side of the first equation, then we have

$$\left(\frac{x}{y}+\frac{y}{x}\right)^2-\frac{3}{2}\left(\frac{x}{y}+\frac{y}{x}\right)=4\frac{3}{50},$$

$$\text{or } z^2-\frac{3}{2}z=\frac{203}{50}\ \therefore z^2-\frac{3}{2}z+\frac{9}{16}=\frac{1849}{400}$$

$$\therefore z-\frac{3}{4}=\pm\frac{43}{20}\ \therefore z=\frac{29}{10}, \text{ or } -\frac{7}{5};$$

$$\text{that is, } \frac{x}{y}+\frac{y}{x}=\frac{29}{10}, \text{ or } -\frac{7}{5}$$

$$\therefore x^2+y^2=\frac{29}{10}xy, \text{ or } x^2+y^2=-\frac{7}{5}xy.$$

From the second equation, $y=\frac{4x-10}{5}$: hence by substitution,

$$x^2+\left(\frac{4x-10}{5}\right)^2=\frac{29x(4x-10)}{50}$$

$$\therefore 25x^2+16x^2-80x+100=58x^2-145x$$

$$\therefore 17x^2-65x=100 \therefore x^2-\frac{65}{17}x=\frac{100}{17}$$

$$\therefore x^2-\frac{65}{17}+\left(\frac{65}{34}\right)^2=\frac{11025}{34^2} \therefore x-\frac{65}{34}=\pm\frac{105}{34}$$

$$\therefore x=5, \text{ or } -\frac{20}{17} \therefore y=\frac{4x}{5}-2=2, \text{ or } -\frac{50}{17}.$$

Another pair of values for x and for y may be obtained from the equation $x^2+y^2=-\frac{7}{5}xy$.

(41.) $\left.\begin{array}{l} x^2-xy=48y. \\ xy-y^2=3x. \end{array}\right\}$ Divide the first by the second, then $\frac{x}{y}=16\frac{y}{x} \therefore x^2=16y^2 \therefore x=\pm 4y.$

Substituting this in the first, $16y^2\mp 4y^2=48y \therefore (16\mp 4)y=48$

$$\therefore y=4, \text{ or } 2\frac{2}{5} \therefore x=16, \text{ or } -9\frac{3}{5}.$$

(42.) $\left.\begin{array}{l} x^2+y^2-15(x+y)=-70. \\ 3xy+31(x+y)=210. \end{array}\right\}$ Multiply the second by $\frac{2}{3}$, then the two equations become

$$x^2+y^2-15(x+y)=-70.\dots \text{(A)}$$

$$2xy+\frac{62}{3}(x+y)=140.\dots \text{(B)}$$

By addition, $(x+y)^2+\frac{17}{3}(x+y)=70.$

Completing the square,

$$(x+y)^2+\frac{17}{3}(x+y)+\left(\frac{17}{6}\right)^2=\frac{2809}{36}$$

$$\therefore x+y+\frac{17}{6}=\pm\frac{53}{6} \therefore x+y=6, \text{ or } -\frac{35}{3}.$$

Taking the first of these, we have from (B),

$$2xy=16 \therefore (x+y)^2-4xy=(x-y)^2=4 \therefore x-y=\pm 2$$
$$\therefore x=4, \text{ or } 2; \; y=2, \text{ or } 4.$$

NOTE.—In every pair of *symmetrical* equations—that is, equations in which x and y may be interchanged without disturbing the equality—it is plain that the resulting values are interchangeable: thus, in the present example, from the values $x=2$, $y=4$, we might *infer* the values $x=4$, $y=2$.

(43.) $\left.\begin{array}{l}\sqrt{y}:\sqrt{x}::\sqrt{x}+3:\sqrt{x}+1.\\ \sqrt{(xy)}+2\sqrt{y}=3x+3\sqrt{x}.\end{array}\right\}$ Converting the proportion into an equation by multiplying extremes and means,

$$\sqrt{xy}+\ \sqrt{y}=\ x+3\sqrt{x}.\dots(A)$$
$$\sqrt{xy}+2\sqrt{y}=3x+3\sqrt{x}.\dots(B)$$

$$\therefore \quad \sqrt{y}=2x \quad \therefore \quad y=4x^2.$$

Hence, by substitution in (A),

$$2x\sqrt{x}+2x=x+3\sqrt{x} \therefore 2x\sqrt{x}+x=3\sqrt{x} \therefore 2x+\sqrt{x}=3$$
$$\therefore x+\tfrac{1}{2}\sqrt{x}+\frac{1}{16}=\frac{25}{16} \therefore \sqrt{x}+\tfrac{1}{4}=\pm\frac{5}{4} \therefore \sqrt{x}=1, \text{ or } -\frac{3}{2}$$
$$\therefore x=1, \text{ or } 2\tfrac{1}{4}; \; y=4x^2=4, \text{ or } 20\tfrac{1}{4}.$$

$$(44.)\ \left.\begin{array}{r}\dfrac{x}{y}-\dfrac{y}{x}=\dfrac{11}{30}\\ x^2+xy=66\end{array}\right\}\therefore\left\{\begin{array}{l}x^2-y^2=\dfrac{11}{30}xy\\ x^2+xy=66.\end{array}\right.$$

These being two homogeneous equations, put vy for x; then

$$v^2y^2-y^2=\frac{11}{30}vy^2 \therefore v^2-1=\frac{11}{30}v \therefore 30v^2-11v=30$$
$$v^2y^2+vy^2=66 \quad \therefore y^2=\frac{66}{v^2+v}.$$

From the former of these,

$$v^2-\frac{11}{30}v+\left(\frac{11}{60}\right)^2=\frac{3721}{3600} \therefore v-\frac{11}{60}=\pm\frac{61}{60} \therefore v=\frac{6}{5}, \text{ or } -\frac{5}{6}$$
$$\therefore y^2=\frac{66}{v^2+v}=25, \text{ or } -\frac{66\times 36}{5} \therefore y=\pm 5, \text{ or } 6\sqrt{-\frac{66}{5}}$$
$$\therefore x=vy=\pm 6, \text{ or } -5\sqrt{-\frac{66}{5}}.$$

(45), (46), (47). These are all pairs of homogeneous equa-

tions, and may be solved in exactly the same way as the pair of equations just discussed.

(48.) $\left.\begin{array}{l} x^{\frac{3}{2}}+y^{\frac{2}{3}}=3x. \\ x^{\frac{1}{2}}+y^{\frac{1}{3}}=x. \end{array}\right\}$ Multiply the second by $x^{\frac{1}{2}}$, then it becomes $x+x^{\frac{1}{2}}y^{\frac{1}{3}}=x^{\frac{3}{2}}$: hence the first is the same as

$$x+x^{\frac{1}{2}}y^{\frac{1}{3}}+y^{\frac{2}{3}}=3x \therefore x^{\frac{1}{2}}y^{\frac{1}{3}}+y^{\frac{2}{3}}=2x;$$

that is, $y^{\frac{1}{3}}(x^{\frac{1}{2}}+y^{\frac{1}{3}})=2x$; but $x^{\frac{1}{2}}+y^{\frac{1}{3}}=x$

$$\therefore y^{\frac{1}{3}}=2 \therefore y=8 \therefore x^{\frac{1}{2}}+2=x$$

$$\therefore x-x^{\frac{1}{2}}=2 \therefore x-\tfrac{1}{2}x^{\frac{1}{2}}+\tfrac{1}{4}=\tfrac{9}{4} \therefore x^{\frac{1}{2}}-\tfrac{1}{2}=\pm\tfrac{3}{2} \therefore x=4, \text{ or } 1.$$

(49.) $\left.\begin{array}{l} \sqrt{\dfrac{3x}{x+y}}+\sqrt{\dfrac{x+y}{3x}}=2. \\ xy-(x+y)=54. \end{array}\right\}$ Squaring the first, $\dfrac{3x}{x+y}+2+\dfrac{x+y}{3x}=4$

$$\therefore \frac{3x}{x+y}+\frac{x+y}{3x}=2 \therefore (3x)^2-2(x+y)3x=-(x+y)^2.$$

Completing the square, $(3x)^2-2(x+y)3x+(x+y)^2=0$

$$\therefore 3x-(x+y)=0 \therefore 2x=y.$$

Hence, by substitution in the second equation,

$$2x^2-3x=54 \therefore x^2-\frac{3}{2}x+\frac{9}{16}=\frac{441}{16}$$

$$\therefore x-\frac{3}{4}=\pm\frac{21}{4} \therefore x=6, \text{ or } -\frac{9}{2}$$

$$\therefore y=2x=12, \text{ or } -9.$$

(50.) $\left.\begin{array}{l} (x+y)=3(x-y)^{\frac{1}{3}}. \\ (x^3+y^3)(x+y)=27. \end{array}\right\}$ Cubing the first, $(x+y)^3=27(x-y)\ldots.$ (A)

From the second, $(x^2-xy+y^2)(x+y)^3=27(x+y)$

By division,

$$x^2-xy+y^2=\frac{x+y}{x-y} \therefore (x^2-xy+y^2)(x-y)=x+y\ldots. \text{ (B)}.$$

Put $x=vy$, in (A) and (B),

$$\therefore (v+1)^3y^3=27(v-1)y \therefore (v+1)^3y^2=27(v-1)$$

$$(v^2y^2-vy^2+y^2)(v-1)y=(v+1)y \therefore (v^2-v+1)y^2=\frac{v+1}{v-1}.$$

Equating the two expressions for y^2, we have

$$\frac{27(v-1)}{(v+1)^3}=\frac{v+1}{(v-1)(v^2-v+1)} \therefore 27(v-1)^2(v^2-v+1)=(v+1)^4$$

$$\therefore\ 26v^4-85v^3+102v^2-85v+26=0$$

$$\therefore\ 26\left(v^2+\frac{1}{v^2}\right)-85\left(v+\frac{1}{v}\right)=-102.$$

Adding 26×2, to each side,

$$26\left(v+\frac{1}{v}\right)^2-85\left(v+\frac{1}{v}\right)=-50.\quad \text{Put } v+\frac{1}{v}=w$$

$$\therefore\ w^2-\frac{85}{26}w+\left(\frac{85}{52}\right)^2=\left(\frac{85}{52}\right)^2-\frac{25}{13}=\frac{2052}{52^2}$$

$$\therefore\ w-\frac{85}{52}=\pm\frac{45}{52}\ \therefore\ w=\frac{5}{2},\text{ or }\frac{10}{13}$$

$$\therefore\ v+\frac{1}{v}=\frac{5}{2},\text{ or } v+\frac{1}{v}=\frac{10}{13}\ \therefore\ v^2-\frac{5}{2}v=-1,\text{ or } v^2-\frac{10}{13}v=-1.$$

From the first of these, $v=2$ or $\frac{1}{2}$;

from the second, $v=\dfrac{5\pm 12\sqrt{-1}}{13}$.

Taking $v=2$, we have $y=\dfrac{27(v-1)}{(v+1)^3}=1\ \therefore\ x=vy=2$;

and these are the values of x and y given in the book. All the other values may be in the same manner obtained from the remaining values of v.

(51.) $\left.\begin{array}{l}\dfrac{x^2+xy+y^2}{x^2-xy+y^2}=2\frac{1}{3}.\\[2ex] \dfrac{1}{x}+\dfrac{1}{y}=1\frac{1}{2}.\end{array}\right\}$ Clearing fractions, the equations are

$$3(x^2+xy+y^2)=7(x^2-xy+y^2)$$
$$2(x+y)=3xy.$$

Put $x=vy$, and they become changed into

$3(v^2y^2+vy^2+y^2)=7(v^2y^2-vy^2+y^2)\ \therefore\ 4v^2-10v+4=0..$ (A)

and $2(v+1)y=3vy^2\ \therefore\ y=\dfrac{2(v+1)}{3v}\ldots.$ (B).

From (A), $v^2-\frac{5}{2}v=-1\ \therefore\ v^2-\frac{5}{2}v+\frac{25}{16}=\frac{9}{16}$

$$\therefore\ v=\frac{5\pm 3}{4}=2\text{ or }\tfrac{1}{2}$$

$\therefore$ (B), $y=1$ or $2\ \therefore\ x=vy=2$ or 1.

(52). $\left.\begin{array}{l}2x^2-2xy=3y.\\ 3xy-3y^2=2x.\end{array}\right\}$ Put $x=vy$, and the equations become

$$2(v^2y^2-vy)=3y \therefore y=\frac{3}{2v(v-1)} \ldots \text{(A)}$$

$$\text{and } 3(vy^2-y^2)=2vy \therefore y=\frac{2v}{3(v-1)} \ldots \text{(B)}$$

$$\therefore \frac{3}{2v(v-1)}=\frac{2v}{3(v-1)} \therefore \frac{3}{2v}=\frac{2v}{3} \therefore 4v^2=9 \therefore v=\pm\tfrac{3}{2}$$

$$\therefore \text{(B)}, y=2, \text{ or } \tfrac{2}{5} \therefore x=vy=3, \text{ or } -\tfrac{3}{5}.$$

(53.) $\left.\begin{array}{l}(x+y)(x-y)^2=32.\\ x^2-y^2-x-y=8.\end{array}\right\}$ Multiplying the second by 4, we have

$$4\{(x^2-y^2)-(x+y)\}=(x+y)(x-y)^2.$$

Dividing by $x+y$,

$$4(x-y-1)=(x-y)^2 \therefore (x-y)^2-4(x-y)=-4$$

$$\therefore (x-y)^2-4(x-y)+4=0 \therefore x-y=2$$

$\therefore$ substituting in the first,

$$4(x+y)=32 \therefore x+y=8 \therefore x=5, y=3.$$

(54.) $\left.\begin{array}{l}4xy^2-x^4y^2=\dfrac{y^6}{4}-4.\\ x^3-xy(x-y)=3.\end{array}\right\}$ From the first, by transposition,

$$y^2\left(x^4+\frac{y^4}{4}\right)=4xy^2+4.$$

Add x^2y^4, then

$$y^2\left(x^4+x^2y^2+\frac{y^4}{4}\right)=x^2y^4+4xy^2+4.$$

Extracting the roots, $y\left(x^2+\dfrac{y^2}{2}\right)=xy^2+2 \ldots \text{(A)}$

$$\therefore x^2y-xy^2=2-\frac{y^3}{2} : \text{ but 2nd equa. } x^2y-xy^2=x^3-3,$$

$$\therefore 2x^3+y^3=10.$$

Subtracting this from 3 times the second equation,

$$x^3-y^3-3xy(x-y)=-1.$$

Extracting the cube root, $x-y=-1 \therefore x=y-1.$

Substituting in (A), $y\left\{(y-1)^2+\dfrac{y^2}{2}\right\}=(y-1)y^2+2$

$$\therefore y(y-1)^2=\frac{y^3}{2}-y^2+2 \therefore y^3-2y^2+2y-4=0$$

$$\text{or } y^2(y-2)+2(y-2)=0.$$

This is satisfied for either $y-2=0$, or $y^2+2=0$

$\therefore y=2$, or $y=\sqrt{-2}$ $\therefore x=y-1=1$, or $-1\pm\sqrt{-2}$.

(55.) $\left.\begin{aligned}(x+y)^{\frac{1}{4}}-(x-y)^{\frac{1}{4}}=a.\\(x+y)^{\frac{1}{4}}+(x-y)^{\frac{1}{4}}=b.\end{aligned}\right\}$ Dividing the first by the second,

$$(x+y)^{\frac{1}{4}}-(x-y)^{\frac{1}{4}}=\frac{a}{b}$$

$$(x+y)^{\frac{1}{4}}+(x-y)^{\frac{1}{4}}=b$$

$$\therefore (x+y)^{\frac{1}{4}}=\tfrac{1}{2}\left(b+\frac{a}{b}\right)$$

$$(x-y)^{\frac{1}{4}}=\tfrac{1}{2}\left(b-\frac{a}{b}\right)$$

$$\therefore \begin{aligned}x+y&=\left\{\tfrac{1}{2}\left(b+\frac{a}{b}\right)\right\}^4\\ x-y&=\left\{\tfrac{1}{2}\left(b-\frac{a}{b}\right)\right\}^4\end{aligned} \quad \therefore \left\{\begin{aligned}x&=\frac{a^2}{4}+\left\{\tfrac{1}{4}\left(b^2+\frac{a^2}{b^2}\right)\right\}^2\\ y&=\frac{a}{4}\left(b^2+\frac{a^2}{b^2}\right)\end{aligned}\right.$$

(56.) $\left.\begin{aligned}(x^2+y^2)(x+y)=2xy.\\(x^4-y^4)(x^2+y^2)=x^2y^2.\end{aligned}\right\}$ Divide the second by the first, then

$$(x-y)(x^2+y^2)=\frac{xy}{2}$$

But $(x+y)(x^2+y^2)=2xy$

$$\therefore \quad 2x(x^2+y^2)=\frac{5xy}{2}\ldots.\text{ (A)}$$

Substituting in the first,

$$\frac{5}{4}(x+y)=2x \therefore 3x=5y \therefore y=\frac{3}{5}x$$

Substituting this in (A),

$$\therefore 34x=\frac{75}{4} \therefore x=\frac{75}{136} \therefore y=\frac{45}{136}.$$

(57.) $\left.\begin{aligned}(x^2+y^2)\frac{y}{x}=\frac{26}{3}.\\(x^2-y^2)\frac{x}{y}=\frac{15}{2}.\end{aligned}\right\}$ By dividing the first by the second,

$$\frac{x^2+y^2}{x^2-y^2}\cdot\frac{y^2}{x^2}=\frac{52}{45} \therefore \frac{x^2+y^2}{x^2-y^2}=\frac{52x^2}{45y^2}$$

$$\text{or } \frac{\frac{x^2}{y^2}+1}{\frac{x^2}{y^2}-1}=\frac{52}{45}\cdot\frac{x^2}{y^2}. \quad \text{Put } \frac{x^2}{y^2}=w, \text{ then } \frac{w+1}{w-1}=\frac{52}{45}w$$

$$\therefore \frac{52}{45}w^2-\frac{52}{45}w=w+1 \;\therefore\; w^2-\frac{97}{52}w=\frac{45}{52}$$

$$\therefore w^2-\frac{97}{52}w+\left(\frac{97}{104}\right)^2=\frac{18769}{104^2}$$

$$\therefore w=\frac{97\pm137}{104}=\frac{9}{4}, \text{ or } -\frac{5}{13} \;\therefore\; \frac{x}{y}=\pm\frac{3}{2}, \text{ or } \sqrt{-\frac{5}{13}}.$$

Taking the first of these values, and substituting in the given equations, we have $x^2+y^2=13$, and $x^2-y^2=5$

$$\therefore x^2=9,\ y^2=4 \;\therefore\; x=\pm3,\ y=\pm2.$$

(58.) $(x^2+y^2)-(x^4+y^4)=\dfrac{63}{256}$.
$(x+y)^2+(xy-2)xy=\dfrac{21}{64}$

The second equation is the same as
$$x^2+y^2+x^2y^2=\frac{21}{64}\,.\,.\ \text{(A)}$$

Subtracting the first from twice this,

$$(x^2+y^2)^2+(x^2+y^2)=\frac{105}{256}$$

$$\therefore (x^2+y^2)^2+(x^2+y^2)+\tfrac{1}{4}=\frac{169}{256}$$

$$\therefore x^2+y^2+\tfrac{1}{2}=\pm\frac{13}{16} \;\therefore\; x^2+y^2=\frac{5}{16}, \text{ or } -\frac{21}{16}.$$

Substituting the first of these in (A),

$$4x^2y^2=\frac{1}{16} \;\therefore\; 2xy=\tfrac{1}{4}.$$

Adding and subtracting,

$$(x+y)^2=\frac{9}{16} \;\therefore\; x+y=\tfrac{3}{4}$$

$$(x-y)^2=\frac{1}{16} \;\therefore\; x-y=\tfrac{1}{4}$$

$$\therefore x=\tfrac{1}{2},\ y=\tfrac{1}{4}.$$

(59.) $(x-2)y+x-2y^2=(y^2-1)\sqrt{xy}$.
$\dfrac{\sqrt{xy}-12}{xy-18}=\dfrac{xy}{4}$.

The first of these is the same as

$$(y+1)x-2y(y+1)=(y^2-1)\sqrt{xy} \therefore x-2y=(y-1)\sqrt{xy}.$$

Put z for $\sqrt{xy}$, then, clearing the second of fractions, we have

$$x-2y=(y-1)z.\ldots\text{(A)}$$
$$4z-48=z^4-18z^2,$$
$$\text{or } z^4-18z^2-4z+48=0,$$
$$\text{or } z^2(z^2-16)-2(z^2+2z-24)=0,$$
$$\text{or } z^2(z^2-4^2)-2(z-4)(z+6)=0,$$

which is satisfied for either

$$z-4=0, \text{ or } z^2(z+4)-2(z+6)=0.$$

The first of these gives $z=4 \therefore xy=16$; and from (A),

$$x-2y=4y-4 \therefore 6y-x=4, \text{ or } \frac{96}{x}-x=4$$

$$\therefore x^2+4x=96 \therefore x^2+4x+4=100 \therefore x+2=\pm 10$$

$$\therefore x=8, \text{ or } -12 \therefore y=\frac{16}{x}=2, \text{ or } -\frac{4}{3}.$$

(60.) $\left.\begin{array}{l}(x^4-y^4)(x^2-y^2)=45x^2y^2.\\(x^2+y^2)(x+y)\ \ =15xy.\end{array}\right\}$ Dividing the first by the second,

$$(x^2-y^2)(x-y)=3xy.\ldots\text{(A)}.$$

Dividing the second by this, $\dfrac{x^2+y^2}{x^2-y^2}\cdot\dfrac{x+y}{x-y}=5$;

$$\text{that is, } \frac{\frac{x^2}{y^2}+1}{\frac{x^2}{y^2}-1}\cdot\frac{\frac{x}{y}+1}{\frac{x}{y}-1}=5, \text{ or } \frac{z^2+1}{z^2-1}\cdot\frac{z+1}{z-1}=5;$$

$$\text{that is, } \frac{z^2+1}{(z-1)^2}=5 \therefore 5(z-1)^2=z^2+1 \therefore 4z^2-10z=-4$$

$$\therefore z^2-\frac{5}{2}z+\frac{25}{16}=\frac{9}{16} \therefore z-\frac{5}{4}=\pm\frac{3}{4} \therefore z=2, \text{ or } \tfrac{1}{2}$$

$$\therefore \frac{x}{y}=2, \text{ or } \tfrac{1}{2} \therefore x=2y, \text{ or } x=\frac{y}{2}.$$

Substituting the first of these values of x in (A), we have

$$3y^3=6y^2 \therefore y=2 \therefore x=2y=4.$$

(61.) $\left.\begin{array}{l}\sqrt[4]{(x+y)}+\sqrt[4]{(x-y)}=y.\\xy+\sqrt{(x^2y^2-y^4)}=\sqrt{(x+y)}+\sqrt{(x-y)}.\end{array}\right\}$ The second equation is the same as

$$\sqrt{(x+y)}+\sqrt{(x-y)}=\frac{y}{2}\{2x+2\sqrt{(x^2-y^2)}\};$$

that is, $\sqrt{(x+y)}+\sqrt{(x-y)}=\frac{y}{2}\{\sqrt{(x+y)}+\sqrt{(x-y)}\}^2$

$$\therefore 1=\frac{y}{2}\{\sqrt{(x+y)}+\sqrt{(x-y)}\}$$

$$\therefore \frac{y}{2}=\frac{1}{\sqrt{(x+y)}+\sqrt{(x-y)}}=\frac{\sqrt{(x+y)}-\sqrt{(x-y)}}{2y}$$

$$\therefore y^2=\sqrt{(x+y)}-\sqrt{(x-y)}$$

$$=\{\sqrt[4]{(x+y)}+\sqrt[4]{(x-y)}\}\{\sqrt[4]{(x+y)}-\sqrt[4]{(x-y)}\}.$$

Substituting for the first of these factors its value in the first equation, we have

$$\sqrt[4]{(x+y)}-\sqrt[4]{(x-y)}=\frac{y^2}{y}=y \therefore \sqrt[4]{(x+y)}=y;$$

$$\text{also } \sqrt[4]{(x-y)}=0 \therefore x=y \therefore 2y=y^4 \therefore y^3=2$$

$$\therefore y=\sqrt[3]{2}, \text{ and } x=\sqrt[3]{2}.$$

(62.) $\left.\begin{array}{l} 8\sqrt{(y+2)}=x+8. \\ x^{\frac{1}{2}}-y^{\frac{1}{2}}=(y+2)^{\frac{1}{2}}. \end{array}\right\}$ Squaring the second, we have

$$x-2\sqrt{xy}+y=y+2$$

$$\therefore (x-2)^2=4xy \therefore y=\frac{(x-2)^2}{4x} \therefore y+2=\frac{(x+2)^2}{4x}.$$

Substituting this in the first,

$$4\frac{x+2}{\sqrt{x}}=x+8=4\sqrt{x}+\frac{8}{\sqrt{x}} \therefore x-4\sqrt{x}+4=\frac{8}{\sqrt{x}}-4$$

$$\therefore (\sqrt{x}-2)^2=\frac{4(2-\sqrt{x})}{\sqrt{x}} \therefore (\sqrt{x}-2)\{\sqrt{x}(\sqrt{x}-2)+4\}=0.$$

This equation is satisfied for either

$$\sqrt{x}-2=0, \text{ or } x-2\sqrt{x}+4=0.$$

The first gives $x=4 \therefore y=\frac{(x-2)^2}{4x}=\frac{1}{4}$.

(63.) $\left.\begin{array}{l} x^2+xy+y^2=\dfrac{133}{x^2-xy+y^2}. \\ x^2-xy+y^2=\dfrac{91}{x^2+y^2}. \end{array}\right\}$ Put $x=vy$, then the equations are

$$(v^2+v+1)y^2=\frac{133}{(v^2-v+1)y^2}$$

$$(v^2-v+1)y^2=\frac{91}{(v^2+1)y^2}.$$

Dividing one by the other, $\dfrac{v^2+v+1}{v^2-v+1}=\dfrac{133(v^2+1)}{91(v^2-v+1)}$

$$\therefore 91(v^2+v+1)=133(v^2+1) \;\therefore\; 42v^2-91v+42=0$$

$$\therefore v^2-\frac{13}{6}v+1=0 \;\therefore\; v^2-\frac{13}{6}v+\left(\frac{13}{12}\right)^2=\frac{25}{144}$$

$$\therefore v=\frac{13\pm5}{12}=\frac{3}{2},\text{ or }\frac{2}{3} \;\therefore\; y=\sqrt[4]{\frac{133}{(v^2+1)^2-v^2}}=2,\text{ or }3$$

$$\therefore x=vy=3,\text{ or }2.$$

(64.) $\left.\begin{aligned}\sqrt{(x^2+\sqrt[3]{x^4y^2})}+\sqrt{(y^2+\sqrt[3]{x^2y^4})}=a.\\ x+y+3\sqrt[3]{bxy}=b.\end{aligned}\right\}$ These equations may be put in the following forms: namely,

$$\sqrt{\{x^{\frac{4}{3}}(x^{\frac{2}{3}}+y^{\frac{2}{3}})\}}+\sqrt{\{y^{\frac{4}{3}}(x^{\frac{2}{3}}+y^{\frac{2}{3}})\}}=a$$

$$(x^{\frac{1}{3}}+y^{\frac{1}{3}})^3=b \;\therefore\; x^{\frac{1}{3}}+y^{\frac{1}{3}}=b^{\frac{1}{3}}\ldots.\text{ (A).}$$

The first is the same as $(x^{\frac{2}{3}}+y^{\frac{2}{3}})\{x^{\frac{2}{3}}+y^{\frac{2}{3}}\}^{\frac{1}{2}}=a$;

that is, $(x^{\frac{2}{3}}+y^{\frac{2}{3}})^{\frac{3}{2}}=a \;\therefore\; x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}\ldots.$ (B).

In order to dispense with fractional exponents, write a, b, x, y, instead of $a^{\frac{1}{3}}$, b , $x^{\frac{1}{3}}$, $y^{\frac{1}{3}}$, and replace them by these at the end: the equations (A) and (B) thus take the simpler forms

$$x+y=b \text{ and } x^2+y^2=a^2$$

$$\therefore x^2+2xy+y^2=b^2$$

$$\therefore x^2-2xy+y^2=2a^2-b^2$$

$$\therefore x-y=\sqrt{(2a^2-b^2)} \;\therefore\; x=\tfrac{1}{2}\{b+\sqrt{(2a^2-b^2)}\}$$

$$y=\tfrac{1}{2}\{b-\sqrt{(2a^2-b^2)}\}.$$

Hence, restoring the symbols to their proper values,

$$x=\tfrac{1}{8}\{b^{\frac{1}{3}}+\sqrt{(2a^{\frac{2}{3}}-b^{\frac{2}{3}})}\}^3,\; y=\tfrac{1}{8}\{b^{\frac{1}{3}}-\sqrt{(2a^{\frac{2}{3}}-b^{\frac{2}{3}}}\}^3.$$

(65.) $\left.\begin{aligned}x^2+y^2=3xy.\\ x^5+y^5=2.\end{aligned}\right\}$ Since $(x+y)^5=x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$,

and from the first equation, $(x+y)^2=5xy$,

and from the second,

$$x^5+y^5=2 \therefore (x+y)^3=x^3+2x^2y+2xy^2+y^3+\frac{2}{5xy};$$

that is, $x^3+3x^2y+3xy^2+y^3=x^3+2x^2y+2xy^2+y^3+\frac{2}{5xy}$

$$\therefore x^2y+xy^2=\frac{2}{5xy} \therefore 5x^2y^2(x+y)=2.$$

But $xy=\frac{(x+y)^2}{5} \therefore (x+y)^5=10 \therefore x+y=\sqrt[5]{10}$

$$\therefore (x+y)^2=(\sqrt[5]{10})^2.$$

Also $4xy=\frac{4}{5}(\sqrt[5]{10})^2$

$$\therefore (x-y)^2=\tfrac{1}{5}(\sqrt[5]{10})^2 \therefore x-y=\frac{\sqrt[5]{10}}{\sqrt{5}}$$

$$\therefore x=\frac{\sqrt[5]{10}}{2\sqrt{5}}(\sqrt{5}+1),\ y=\frac{\sqrt[5]{10}}{2\sqrt{5}}(\sqrt{5}-1).$$

(66.) $\left.\begin{array}{l}(x+y)^3=64(x-y).\\(x^3+y^3)(x+y)=76.\end{array}\right\}$ Put $x=vy$, then the equations are

$$(v+1)^3y^3=64(v-1)y \therefore y^2=\frac{64(v-1)}{(v+1)^3}\cdots\cdots(A)$$

$$(v^3+1)(v+1)y^4=76 \therefore y^4=\frac{76}{(v^3+1)(v+1)}\cdots\cdots(B)$$

$$\therefore \frac{64^2(v-1)^2}{(v+1)^6}=\frac{76}{(v^3+1)(v+1)} \therefore \frac{64^2(v-1)^2}{(v+1)^4}=\frac{76}{v^2-v+1}$$

$$\therefore 64^2(v-1)^2(v^2-v+1)=76(v+1)^4.$$

Dividing by 4, and transposing,

$$1005v^4-3148v^3+3982v^2-3148v+1005=0$$

$$\therefore \left(v^2+\frac{1}{v^2}\right)-\frac{3148}{1005}\left(v+\frac{1}{v}\right)=-\frac{3982}{1005}.$$

Adding 2, $\left(v+\frac{1}{v}\right)^2-\frac{3148}{1005}\left(v+\frac{1}{v}\right)=-\frac{1972}{1005}$

$$\text{or } w^2-\frac{3148}{1005}w+\left(\frac{1574}{1005}\right)^2=\frac{495616}{1005^2}$$

$$\therefore w-\frac{1574}{1005}=\pm\frac{704}{1005}$$

$$\therefore w=\frac{2278}{1005}, \text{ or } \frac{58}{67}.$$

Taking the first of these,

$$v+\frac{1}{v}=\frac{2278}{1005} \therefore v^2-\frac{2278}{1005}v=-1$$

$$\therefore v^2-\frac{2278}{1005}v+\left(\frac{1139}{1005}\right)^2=\frac{287296}{1005^2}$$

$$\therefore v-\frac{1139}{1005}=\pm\frac{536}{1005} \therefore v=\frac{5}{3}, \text{ or } \frac{201}{335}.$$

Taking the first of these, we have (A)

$$y=8\frac{\sqrt{(v-1)}}{(v+1)\sqrt{(v+1)}}=\frac{3}{2} \therefore x=vy=\frac{5}{2}.$$

(67.) $\left.\begin{array}{l} x^4+y^4=1+2xy+3x^2y^2. \\ x^3+y^3=2y^2x+2y^2+x+1. \end{array}\right\}$ From the first equation by transposing, we have

$$x^4-2x^2y^2+y^4=1+2xy+x^2y^2 \therefore x^2-y^2=1+xy.\dots (A)$$

$$\therefore x^2-xy+y^2=2y^2+1.$$

But the second is $x^3+y^3=(2y^2+1)(x+1)$

$$\therefore x^3+y^3=(x^2-xy+y^2)(x+1)$$

Dividing by x^2-xy+y^2, $x+y=x+1 \therefore y=1 \therefore$ (A), $x^2-x=2$

$$\therefore x^2-x+\frac{1}{4}=\frac{9}{4} \therefore x=\frac{1\pm 3}{2}=2, \text{ or } -1.$$

(68.) $\left.\begin{array}{l} (x^2+xy+y^2)(x^2-xy+y^2)=481. \\ (x^2+y^2)^2-(x^2+y^2)xy=325. \end{array}\right\}$ Put v for x^2+y^2, and w for xy; then the equations are changed into

$$\left.\begin{array}{l} (v+w)(v-w)=481 \\ \text{and } (v-w)v=325 \end{array}\right\} \therefore \frac{v+w}{v}=\frac{481}{325}=\frac{37}{25} \therefore \frac{w}{v}=\frac{12}{25};$$

$$\therefore w=\frac{12}{25}v \therefore v^2-w^2=\left(1-\frac{12^2}{25^2}\right)v^2=\frac{481v^2}{25^2}=481$$

$$\therefore v=25 \therefore w=12;$$

that is, $x^2+y^2=25$, $xy=12 \therefore x^2+2xy+y^2=49$
$x^2-2xy+y^2=\ 1$

$$\therefore x+y=\pm 7,\ x-y=\pm 1 \therefore x=4,\ y=3,$$

or $x=3$, $y=4$, or $x=-4$, $y=-3$, or $x=-3$, $y=-4$.

NOTE.—It is plain, from the symmetrical form of each equation, that x and y may be interchanged; and that whatever pair of values answer, the same pair with changed signs will also answer.

(69.) $\left.\begin{array}{l} x-2\sqrt{(2ay-y^2)}=5\sqrt{ay}. \\ x^2-8x\sqrt{ay}=2ay-9y^2. \end{array}\right\}$ Add $16ay$ to each side of the second equation, then it becomes

$$(x-4\sqrt{ay})^2=9(2ay-y^2) \therefore x-4\sqrt{ay}=3\sqrt{(2ay-y^2)}.$$

Hence by substitution the first is

$$\frac{x+8\sqrt{ay}}{3}=5\sqrt{ay} \therefore x=7\sqrt{ay} \therefore x-7ay=0.$$

Subtracting this from the equation above, we have

$$3\sqrt{ay}=3\sqrt{(2ay-y^2)} \therefore ay=2ay-y^2 \therefore y=a \therefore x=7a.$$

(70.) $\left.\begin{array}{l} x^6+y^6=\dfrac{1}{\sqrt{2}}. \\ x^4+y^4=1. \end{array}\right\}$ Put $x^2=z+v$, and $y^2=z-v$, then the equations are

$$(z+v)^3+(z-v)^3=\frac{1}{\sqrt{2}}$$

$$(z+v)^2+(z-v)^2=1\,;$$

$$\text{that is, } 2z^3+6zv^2=\frac{1}{\sqrt{2}} \therefore v^2=\frac{1-2z^3\sqrt{2}}{6z\sqrt{2}}$$

$$2z^2+2v^2=\ 1 \ \therefore v^2=\frac{1-2z^2}{2}$$

$$\therefore \frac{1-2z^3\sqrt{2}}{6z\sqrt{2}}=\frac{1-2z^2}{2}, \text{ that is, } \frac{1-(z\sqrt{2})^3}{3z\sqrt{2}}=1-(z\sqrt{2})^2.$$

Divide by $1-z\sqrt{2}$,

$$\therefore 1+z\sqrt{2}+(z\sqrt{2})^2=(1+z\sqrt{2})3z\sqrt{2}\ldots.\ (A).$$

Since the equation is thus divisible by $1-z\sqrt{2}$, it is satisfied

for $1+z\sqrt{2}=0 \therefore z=\dfrac{1}{\sqrt{2}}$ (see NOTE, p. 56,)

$$\therefore v=\sqrt{\frac{1-2z^2}{2}}=0 \therefore x=\sqrt{(z+v)}=\frac{1}{\sqrt[4]{2}},\ y=\sqrt{(z-v)}=\frac{1}{\sqrt[4]{2}}$$

These are the values in the book: other values are obtained from the equation (A), which gives $z=\dfrac{-1\pm\sqrt{3}}{\sqrt{2}}$.

(71,) $\left.\begin{array}{l} x^7-y^7=127. \\ x-y=1. \end{array}\right\}$ Raising the second to the seventh power, subtracting the result from the first, and dividing by $7xy$, we have

$$x^5-3x^4y+\ 5x^3y^2-\ 5x^2y^3+3xy^4-y^5=\frac{18}{xy}$$

But $(x-y)^5=x^5-5x^4y+10x^3y^2-10x^2y^3+5xy^4-y^5=1$

$$\therefore\ 2x^4y-5x^3y^2+5x^2y^3-2xy^4=\frac{18}{xy}-1$$

$$\therefore\ 2x^3-5x^2y+5xy^2-2y^3=\frac{18-xy}{(xy)^2}$$

Also $2(x-y)^3=2x^3-6x^2y+6xy^2-2y^3=2$

$$\therefore\ x^2y-xy^2=\frac{18-xy-2(xy)^2}{(xy)^2}$$

But $x-y=1\ \therefore\ x^2y-xy^2=(x-y)xy=xy$

$$\therefore\ (xy)^3+2(xy)^2+xy-18=0.$$

This is evidently satisfied for $xy=2$; and since $(x-y)^2=1$,

$\therefore\ (x-y)^2+4xy=(x+y)^2=9\ \therefore\ x+y=3\ \therefore\ x=2,\ y=1.$

(72.) $\left.\begin{array}{r} x+y+z=3a. \\ xy+xz+yz=3a^2. \\ xyz=a^3. \end{array}\right\}$ Multiply the second by x, and subtract from the third: $\therefore\ -x^2(y+z)=a^3-3a^2x.$

Multiply the first by x^2, and add to this, then

$$x^3=a^3-3a^2x+3ax^2\ \therefore\ a^3-3a^2x+3ax^2-x^3=0;$$

that is, $(a-x)^3=0\ \therefore\ x=a.$

In like manner, by employing y instead of x in the foregoing steps, we shall have $y=a$, and by using z, $z=a$.

(73.) $\left.\begin{array}{l} xy=x+y. \\ xz=2(x+z). \\ yz=3(y+z). \end{array}\right\}$ By division, these equations become converted into the following: namely,

$$\frac{1}{x}+\frac{1}{y}=1,\quad \frac{1}{x}+\frac{1}{z}=\frac{1}{2},\quad \frac{1}{y}+\frac{1}{z}=\frac{1}{3}$$

$$\therefore\ \frac{1}{y}-\frac{1}{z}=\frac{1}{2},\ \text{also}\ \frac{1}{x}-\frac{1}{y}=\frac{1}{6},\ \text{and}\ \frac{1}{x}-\frac{1}{z}=\frac{2}{3}$$

$$\therefore\ \frac{2}{x}=\frac{7}{6},\quad \frac{2}{y}=\frac{5}{6},\quad \text{and}\quad \frac{2}{z}=-\frac{1}{6}$$

$$\therefore\ x=1\frac{5}{7},\quad y=2\frac{2}{5},\quad z=-12.$$

(74.) $\left.\begin{array}{l} x^2+y^2+z^2=14. \\ x^2+y=3. \\ z^2+y=11. \end{array}\right\}$ Subtracting the second from the third, $z^2-x^2=8.$

$$\left.\begin{array}{l}\text{Adding to the first, } 2z^2+y^2=22\\ \text{Twice the third, } 2z^2+2y=22\end{array}\right\} \therefore y^2-2y=0 \therefore y=2$$

$$\therefore x=\sqrt{(3-y)}=\pm 1,\ z=\sqrt{(11-y)}=\pm 3.$$

(75.) $\left.\begin{array}{l}xy+z=5.\\ xyz=4.\\ 2(x^2-y)=(y^2-x)^2.\end{array}\right\}$ Multiplying the first by z, and subtracting the second, we have

$$z^2=5z-4 \therefore z^2-5z=-4 \therefore z=1, \text{ or } 4 \therefore xy=4, \text{ or } 1.$$

Substituting $\frac{4}{x}$ for y in the third equation,

$$2x^2-\frac{8}{x}=\frac{16^2}{x^4}-\frac{32}{x}+x^2 \therefore x^2+\frac{24}{x}=\frac{16^2}{x^4}$$

$$\therefore x^6+24x^3=256 \therefore x^6+24x^3+144=400 \therefore x^3+12=\pm 20$$

$$\therefore x=2 \therefore y=\frac{4}{x}=2, \text{ or } x=-2\sqrt[3]{4},\ y=-\sqrt[3]{2}.$$

Substituting $\frac{1}{x}$ for y in the third equation,

$$2x^2-\frac{2}{x}=\frac{1}{x^4}-\frac{2}{x}+x^2 \therefore x^2=\frac{1}{x^4} \therefore x=\frac{1}{x^2} \therefore x=\pm 1 \therefore y=\pm 1.$$

Hence the values are $\begin{array}{l}x=2,\ -2\sqrt[3]{4},\ \pm 1\\ y=2,\ -\sqrt[3]{2},\ \pm 1\end{array}$ $z=1, 4.$

(76.) $\left.\begin{array}{l}x^2+xy+y^2=13.\\ y^2+yz+z^2=49.\\ x^2+xz+z^2=31.\end{array}\right\}$ Subtracting the third from the second, and the first from the third, we have

$$(y^2-x^2)+2(y-x)=18$$
$$(z^2-y^2)+2(z-y)=18$$

that is, $(y-x)(x+y+z)=18.\dots$ (A)

$(z-y)(x+y+z)=18.\dots$ (B)

$\therefore (z-x)(x+y+z)=36.\dots$ (C)

$$\therefore \text{(A), (B), } y-x=z-y \therefore z+x=2y$$

$$\therefore \text{(C), } (z-x)3y=36 \therefore z-x=\frac{12}{y}$$

$$\therefore z=y+\frac{6}{y},\ x=y-\frac{6}{y}.$$

Substituting these in the third equation,

$$\left(y-\frac{6}{y}\right)^2+y^2-\frac{36}{y^2}+\left(y+\frac{6}{y}\right)^2=31;$$

$$\text{that is, } 3y^2+\frac{36}{y^2}=31 \therefore 3y^4-31y^2=-36$$

$$\therefore y^4-\frac{31}{3}y^2+\left(\frac{31}{6}\right)^2=\frac{529}{36}$$

$$\therefore y^2-\frac{31}{6}=\pm\frac{23}{6} \therefore y^2=9, \text{ or } \frac{4}{3} \therefore y=\pm 3, \text{ or} \pm\frac{2}{\sqrt{3}}$$

$$\therefore x=y-\frac{6}{y}=1, \text{ or } -1;\ z=y+\frac{6}{y}=5, \text{ or } -5$$

$$\therefore x=\pm 1,\ y=\pm 3,\ z=\pm 5.$$

Another set of pairs may be obtained from $y=\pm\frac{2}{\sqrt{3}}$.

(77.) $\left.\begin{array}{l} x^{-2}y^{-1}z=1\frac{1}{2}. \\ x^{-1}y\,z^2=18. \\ xy^2z^3=108. \end{array}\right\}$ Divide the second by the first, and the third by the second, then we have

$$xy^2z=12,\ x^2yz=6$$

$$\therefore xy^2z=2x^2yz \therefore y=2x \therefore z=\frac{3}{x^3}.$$

Substituting these in the third equation,

$$x\,.\,4x^2\,.\,\frac{27}{x^9}=108 \therefore \frac{1}{x^6}=1 \therefore x=\pm 1 \therefore y=\pm 2,\ z=\pm 3.$$

(78.) This does not appear to admit of solution by quadratics.

Problems (Page 97).

(1.) Let x be the smaller number, then $x+8$ is the greater; and by the question,

$$x^2+8x=128 \therefore x^2+8x+16=144$$

$$\therefore x+4=\pm 12 \therefore x=8, \text{ or } -16 \therefore x+8=16, \text{ or } -8.$$

Hence the numbers are either 8 and 16, or -16 and -8.

(2.) Let x, y represent the numbers; then by the question,

$$x+y=40, \text{ and } x^2+y^2=818.$$

Squaring the first equation, $x^2+2xy+y^2=1600$.

Subtracting, $2xy=782 \therefore x^2-2xy+y^2=36 \therefore x-y=\pm 6$.

$$\text{But } x+y=40 \therefore x=\frac{40\pm6}{2}, \text{ and } y=\frac{40\mp6}{2};$$

that is, $x=23$, or 17; $y=17$ or 23.

NOTE.—As observed at p. 115, the values of x in a pair of *symmetrical* equations, when interchanged, always give those of y; so that x in this example having been found to be 23 or 17, we might have inferred that y is 17 or 23.

(3.) Let x be the magnitude, then we are to have the condition,

$$x=\frac{1}{x}+1 \therefore x^2-x=1 \therefore x^2-x+\tfrac{1}{4}=\frac{5}{4}$$

$$\therefore x-\tfrac{1}{2}=\tfrac{1}{2}\sqrt{5} \therefore x=\tfrac{1}{2}(1\pm\sqrt{5}).$$

(4.) Suppose he bought x sheep, then the cost price, in shillings, of each, was $\frac{1200}{x}$; and the selling price of each of the $x-15$ disposed of was $\frac{1080}{x-15}$ shillings: hence by the question,

$$\frac{1200}{x}+2=\frac{1080}{x-15} \therefore \text{clearing, } 2x^2+90x=18000$$

$$\therefore x^2+45x=9000 \therefore x^2+45x+\left(\frac{45}{2}\right)^2=\frac{38025}{4}$$

$$\therefore x+\frac{45}{2}=\pm\frac{195}{2} \therefore x=75, \text{ or } -120.$$

Hence the number of sheep was 75; the -120, although satisfying the algebraical condition, being excluded by the nature of the question.

(5.) Suppose they finish the work in x hours; then by the question, A alone could have done it in $x+6$ hours

B " " " $x+15$ "

C " " " $2x$ "

Hence the parts of the whole they can severally do in one hour are

$$\frac{1}{x+6}, \quad \frac{1}{x+15}, \quad \text{and } \frac{1}{2x}.$$

Consequently, as x times the sum of these parts make up the whole work, we must have

$$\frac{x}{x+6}+\frac{x}{x+15}+\frac{1}{2}=1, \text{ or } \frac{x}{x+6}+\frac{x}{x+15}=\frac{1}{2}.$$

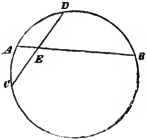
Clearing fractions, and transposing,

$$3x^2+21x=90 \therefore x^2+7x=30 \therefore x^2+7x+\frac{49}{4}=\frac{169}{4}$$

$$\therefore x+\frac{7}{2}=\pm\frac{13}{2} \therefore x=3, \text{ or } -10.$$

Hence the time occupied is 3 hours.

(6.) By Prop. 35, Book III., of Euclid,



$$AE \times EB = CE^2 = \frac{CD^2}{4}.$$

By the question, $AE=25$,
and $EB=\frac{4}{5}CD-16$.

Hence, representing the line CD by x, we have

$$25\left(\frac{4}{5}x-16\right)=\frac{x^2}{4},$$

$$\text{or } 20x-400=\frac{x^2}{4}$$

$$\therefore x^2-80x=-1600 \therefore x^2-80x+1600=0 \therefore x=40.$$

Hence the length of the path CD is 40 feet.

(7.) Suppose he bought x oxen, then the price of each was $\frac{240}{x}$ pounds, also the price received for each of the $x-3$ sold was $\frac{240}{x}+8$; and therefore the money received for all of them was

$$(x-3)\left(\frac{240}{x}+8\right)=240+59 \text{ by the question;}$$

$$\text{that is, } \frac{240(x-3)}{x}+8x-24=299,$$

$$\text{or } 240-\frac{720}{x}+8x-24=299$$

$$\therefore\ 8x^2-83x=720\ \therefore\ x^2-\frac{83}{8}x+\left(\frac{83}{16}\right)^2=90+\left(\frac{83}{16}\right)^2=\frac{29929}{16^2}$$

$$\therefore\ x-\frac{83}{16}=\pm\frac{173}{16}\ \therefore\ x=\frac{83\pm173}{16}=16,\text{ or }-\frac{45}{8}.$$

Hence the number bought was 16.

(8.) Let x be the number of persons, then by the question the share of each would have been $\frac{72}{x}$ shillings; but it appears that each of the $x-2$ persons paid $\frac{72}{x}+\frac{1}{2}$ shillings.

$$\therefore\ (x-2)\left(\frac{72}{x}+\frac{1}{2}\right)=72\ \therefore\ \frac{72(x-2)}{x}+\frac{x}{2}=73$$

$$\therefore\ x^2-2x=288\ \therefore\ x^2-2x+1=289\ \therefore\ x-1=\pm17$$

$$\therefore\ x=18,\text{ or }-16.$$

Hence the number of persons was 18.

(9.) Suppose he began with x shillings: then at the end of the first game he had $3x-16$, and at the end of the second,

$$\tfrac{1}{3}(3x-16)+x=80,\text{ by the question,}$$

$$\therefore\ 3x-16+5x=400\ \therefore\ 8x=416\ \therefore\ x=52.$$

Consequently, he began with $52s$.

(10.) Suppose the breadth to be x yards, then the length must be $x+10$ yards; and since the length multiplied by the breadth is the numerical value of the surface, we have the equation

$$x^2+10x=3000\ \therefore\ x^2+10x+25=3025$$

$$\therefore\ x+5=\pm55\ \therefore\ x=50,\text{ or }-60\ \therefore\ x+10=60,\text{ or }-50.$$

Hence the length is 60 yards, and the breadth 50 yards.

(11.) Suppose A put in x pounds, then as this sum was in trade 12 months it is the same as if $12x$ pounds had been put in for one month. In like manner, B's contribution is the same as £480 for one month: and since the whole stock is to the whole gain as A's contribution to that stock is to his share of the gain, we have the proportion

$12x+480 : 12x :: 18 :$ A's gain, or $x+40 : x :: 18 :$ A's gain; therefore, by the Rule of Three,

$$\frac{18x}{x+40} = \text{A's gain} = 26 - x \text{ by the question}$$
$$\therefore 18x = (x+40)(26-x) = 1040 - 14x - x^2$$
$$\therefore x^2 + 32x = 1040 \therefore x^2 + 32x + 256 = 1296$$
$$\therefore x + 16 = \pm 36 \therefore x = 20, \text{ or } -52.$$
Consequently, A contributed £20.

(12.) Let the side of the smaller square be x feet, then the side of the other is $x+12$ feet: the number of square feet in these is x^2 and $(x+12)^2$, so that the number of square stones in the two is $x^2 + (x+12)^2 = 2120$ by the question: therefore,
$$2x^2 + 24x + 144 = 2120 \therefore x^2 + 12x = 988$$
$$\therefore x^2 + 12x + 36 = 1024 \therefore x + 6 = \pm 32 \therefore x = 26 \therefore x + 12 = 38.$$
Therefore, the sides of the squares are 26 feet and 38 feet respectively.

(13.) Let x be the number of pounds the horse cost: then the gain was $56-x$, which by the question is a gain of x per cent.: that is,
$$x : 56 - x :: 100 : x \therefore x^2 = 5600 - 100x$$
$$\therefore x^2 + 100x = 5600 \therefore x^2 + 100x + 2500 = 8100$$
$$\therefore x + 50 = \pm 90 \therefore x = 40, \text{ the number of pounds.}$$

(14.) Let the two numbers be x and y: then by the question,
$$x - y = 6 \therefore (x-y)^2 = 36 \therefore 2xy = x^2 + y^2 - 36 \ldots \text{(A)}$$
$$xy(x^2+y^2) = 4640 \therefore (x^2+y^2-36)(x^2+y^2) = 9280;$$
$$\text{that is, } (x^2+y^2)^2 - 36(x^2+y^2) = 9280$$
$$\therefore (x^2+y^2)^2 - 36(x^2+y^2) + 18^2 = 9604$$
$$\therefore x^2 + y^2 - 18 = \pm 98 \therefore x^2 + y^2 = 116 \ldots \text{(B)}$$
$$\therefore \text{(A)}, 2xy = 80 \therefore \text{(B)}, x + y = \sqrt{196} = \pm 14.$$
And since $x - y = 6 \therefore x = 10$ and $y = 4$; or $x = -4$, $y = -10$.

(15.) Suppose they travel x miles an hour and y miles an hour respectively: then they will have finished the journey in
$$\frac{150}{x} \text{ hours and } \frac{150}{y} \text{ hours respectively;}$$
$$\text{and by the question, } x = y + 3, \text{ and } \frac{150}{x} + 8\tfrac{1}{3} = \frac{150}{y}.$$

$$\text{Clearing, } 450y+25xy=450x, \text{ or putting } y+3 \text{ for } x,$$
$$450y+25y^2+75y=450y+1350 \therefore 25y^2+75y=1350$$
$$\therefore y^2+3y=54 \therefore y^2+3y+\frac{9}{4}=\frac{225}{4}$$
$$\therefore y+\tfrac{3}{2}=\pm 7\tfrac{1}{2} \therefore y=6 \therefore x=y+3=9.$$

Hence A travels 9 miles an hour, and B 6 miles.

(16.) Suppose x to be the portion of the whole the man can do in 1 hour, and y the portion the boy can do in 1 hour: then by the question,

$$10x+6y=1, \text{ and } 6x+10y=\tfrac{2}{3}.$$

$$\left.\begin{array}{ll}\text{Subtracting, } & 4x - 4y=\tfrac{1}{3} \therefore x-y=\dfrac{1}{12} \\ \text{Adding, } & 16x+16y=\tfrac{5}{3} \therefore x+y=\dfrac{}{48}\end{array}\right\} \therefore x=\frac{9}{96}, y=\frac{1}{96}$$

These are the portions done in 1 hour by the man and boy respectively. Hence, 1 hour's work of the man is worth 9 hours' work of the boy. Suppose, now, the man to work z hours, and therefore the boy $z-5$ hours: then

$$\frac{9z}{96}+\frac{z-5}{96}=1 \therefore 10z=101 \therefore z=10\tfrac{1}{10} \therefore z-5=5\tfrac{1}{10}.$$

Therefore, the man works $10\frac{1}{10}$ hours, and the boy $5\frac{1}{10}$ hours.

(17.) Suppose the stream runs x miles an hour, and that the rower could go y miles an hour in still water: then when assisted by the stream, he goes $y+x$ miles an hour, and when opposed by it, $y-x$ miles an hour; so that he goes the 20 miles in the one case in $\dfrac{20}{y+x}$ hours, and in the other case in $\dfrac{20}{y-x}$ hours: these times are by the question as 2 to 3; that is,

$$\frac{60}{y+x}=\frac{40}{y-x} \therefore \frac{3}{y+x}-\frac{2}{y-x}=0\ldots. \text{(A)}$$

$$\text{And moreover } \frac{20}{y+x}+\frac{20}{y-x}=10 \therefore \frac{2}{y+x}+\frac{2}{y-x}=1\ldots. \text{(B)}$$

Adding these equations, we have

$$\frac{5}{y+x}=1 \therefore y+x=5 \therefore \text{(A)}, y-x=\frac{10}{3} \therefore y=\frac{25}{6}, x=\frac{5}{6}.$$

$$\text{Also, } \frac{20}{y+x}=4, \text{ and } \frac{20}{y-x}=6;$$

hence the stream runs $\frac{5}{6}$ mile an hour: the time of going with the stream is 4 hours, and of returning 6 hours.

(18.) Suppose there were x men in the front of the wedge, then in the next row behind there were $x-1$, and in the third row $x-2$; behind this row the space was vacant.

In like manner, for the three rows in either of the other sides of the wedge, we should have x, $x-1$, and $x-2$ men. But it is plain that each corner of the wedge—composed of 9 men—is thus counted *twice;* so that three times 9 or 27 must be deducted; that is,

$$3\{x+(x-1)+(x-2)-9\}, \text{ or } 9x-36$$

is the whole number of men. Hence the number of men forming the hollow square is $9x-36-597$, or $9x-633$. The number of men in the four rows of each side of this square would be the fourth part of $9x-633$, were it not that each corner of the square—composed of 16 men—would thus be reckoned *twice;* consequently, if four times 16 be added to the entire number of men, *then* a fourth part of the whole will correctly express the number in each of the sides, and therefore a fourth of this the number in the front row.

$$\therefore \frac{9x-633+64}{16}=\frac{9x-569}{16}=\sqrt{x}+1 \text{ by the question,}$$

$$\therefore \frac{9x-585}{16}=\sqrt{x} \;\therefore\; 9x-585=16\sqrt{x} \;\therefore\; 9x-16\sqrt{x}=585$$

$$\therefore x-\frac{16}{9}\sqrt{x}+\frac{64}{81}=\frac{5329}{81} \;\therefore\; \sqrt{x}-\frac{8}{9}=\pm\frac{73}{9}$$

$$\therefore \sqrt{x}=9, \;\therefore\; x=81 \;\therefore\; 9x-36=693.$$

Hence the number of men in the wedge was 693.

(19.) Let x be the number of pounds contributed by A, then $100-x$ is the number contributed by B. The profit awarded to A is $99-x$, and to B, $99-100+x$, by the question; also (see solution to Question 11),

$$3x+2(100-x) : 98 \text{ (sum of profits)} :: 3x : 99-x, \text{ A's profit;}$$

$$\text{that is, } x+200 : 98 :: 3x : 99-x$$

$$\therefore (x+200)(99-x)=294x \;\therefore\; x^2+395x=19800$$

$$\therefore x^2+395x+\left(\frac{395}{2}\right)^2=\frac{235225}{4}$$

$$\therefore x+\frac{395}{2}=\pm\frac{485}{2} \therefore x=45 \therefore 100-x=55.$$

Hence the contributions were £45 and £55.

(20.) Suppose the rates of marching were x and $x+\frac{1}{4}$ miles an hour; then the respective times of completing the 39 miles are $\frac{39}{x}$ and $\frac{39}{x+\frac{1}{4}}$ hours; and by the question,

$$\frac{39}{x}-1=\frac{39}{x+\frac{1}{4}}, \text{ or } \frac{39}{x}-1=\frac{156}{4x+1}$$

$$\therefore 156x+39-4x^2-x=156 \therefore 4x^2+x=39$$

$$\therefore x^2+\frac{x}{4}+\frac{1}{64}=\frac{39}{4}+\frac{1}{64}=\frac{625}{64}$$

$$\therefore x+\frac{1}{8}=\pm\frac{25}{8} \therefore x=3 \therefore x+\tfrac{1}{4}=3\tfrac{1}{4}.$$

Therefore the rates were 3 and $3\frac{1}{4}$ miles an hour.

(21.) Suppose a dozen of sherry cost x pounds: then for £10 there were $\frac{10}{x}$ dozens; and, by the question, for £6 there were $\frac{10}{x}-3$ dozens of claret, so that the price of 1 dozen was $6\div\left(\frac{10}{x}-3\right)$, or $\frac{6x}{10-3x}$. Consequently, by the question,

$$7x+\frac{72x}{10-3x}=50 \therefore 70x-21x^2+72x=500-150x$$

$$\therefore 21x^2-292x=-500$$

$$\therefore x^2-\frac{292}{21}x+\left(\frac{146}{21}\right)^2=\frac{21316}{21^2}-\frac{500}{21}=\frac{10816}{21^2}$$

$$\therefore x-\frac{146}{21}=\pm\frac{104}{21} \therefore x=2 \text{ or } \frac{250}{21} \therefore \frac{6x}{10-3x}=3.$$

Hence the sherry was £2 per dozen, and the claret £3.

(22.) The first arrangement of the troops was evidently into two squares, since there were as many ranks in each as men in front. Let x be a side of one square, and y a side of

the other: then x^2+y^2 expresses the number of men. In the second arrangement, the men in the square A or x^2 had y men in front; hence, the number of ranks was $\frac{x^2}{y}$; in like manner, the number of ranks in the second arrangement of the square B or y^2 was $\frac{y^2}{x}$: by the question, both these numbers together make 91: hence we have these two equations; namely,

$$x+y=84, \quad \frac{x^2}{y}+\frac{y^2}{x}=91, \text{ or } x^3+y^3=91xy.$$

Substituting $84-y$ for x, $(84-y)^3+y^3=91y(84-y)$;

that is, $84^3-3 \,.\, 84^2y+3 \,.\, 84y^2=91 \,.\, 84y-91y^2$.

Or dividing by 7,

$$12 \,.\, 84^2-3 \,.\, 12 \,.\, 84y+3 \,.\, 12y^2=13 \,.\, 84y-13y^2.$$

Transposing, $49y^2-49 \,.\, 84y=-12 \,.\, 84^2 \therefore y^2-84y=-12^3$

$$\therefore y^2-84y+42^2=42^2-12^3=36$$

$$\therefore y-42=\pm 6 \therefore y=36, \text{ or } 48 \therefore x=84-y=48, \text{ or } 36.$$

Consequently, the number of men in a side of the square A is 36, and the number in the side of the square B is 48; or the conditions will be equally satisfied if 48 be the number in the side of A, and 36 the number in the side of B: the number of men in the square columns are, therefore, 36^2 and 48^2; that is, 1296 and 2304.

(23.) Let x represent the number of pounds in A's capital: then, by the question, A gains $11-x$, which is at the rate of $\frac{100(11-x)}{x}$ per cent. This, therefore, expresses B's capital; and since £36 is his gain, B's gain per cent. must be $3600 \div \frac{100(11-x)}{x}=\frac{36x}{11-x}$; so that, by the question,

$$\frac{100(11-x)}{x}=\frac{144x}{11-x} \therefore 100(11-x)^2=144x^2$$

$$\therefore 10(11-x)=12x \therefore 22x=110 \therefore x=£5, \text{ A's capital}$$

$$\therefore \frac{100(11-x)}{x}=£120, \text{ B's capital.}$$

(24.) Let x represent the number of men in front at first, then the number in depth was $x+5$. But after the appear-

ance of the enemy the number in front was $x+845$, and the number in depth 5. The total number of men being the same in both arrangements, we have

$$x(x+5)=5(x+845) \therefore x^2=4225 \therefore x=65$$

$$\therefore x(x+5)=65 \,.\, 70=4550, \text{ the number of men.}$$

(25.) Let $6x$ represent the whole number of gallons, then by the question there were $3x+25$ gallons of wine,

and $2x-5$,, cider

$$\therefore 6x=5x+20 \therefore x=20.$$

Therefore, there were $3x+25=85$ gallons of wine, and $2x-5=35$ gallons of cider.

(26.) Suppose the number of feet in the base to be x; then the number in the hypotenuse is $x+6$. And therefore (Euc., 47 of I.)

$$(x+6)^2-x^2=\text{perp.}^2=(x+6-3)^2=(x+3)^2;$$

that is, $12x+36=12(x+3)=(x+3)^2 \therefore 12=x+3 \therefore x=9$. Hence, the base is 9, the perpendicular $x+3=12$, and the hypotenuse $x+6=15$.

(27.) Suppose the left hand digit to be x, and the right hand one y; then the number is expressed by $10x+y$; and, by the question, we have the equations,

$$(10x+y)x=46, \text{ and } (x+y)x=10 \ldots . \text{ (A)}.$$

Subtracting the second from the first, $9x^2=36 \therefore x=2$

$$\therefore \text{(A)}, 2(2+y)=10 \therefore y=\frac{6}{2}=3: \text{ hence the number is 23.}$$

(28.) To avoid fractions, suppose A's stock was $40x$, then by the question, $20x+100$ was B's stock, and $3x+15$ was A's gain; consequently, $100-(3x+15)=85-3x$ was B's gain. And as the gains are as the stocks contributed, we have the proportion

$$40x : 20x+100 :: 3x+15 : 85-3x,$$

$$\text{or } 2x : x+5 :: 3x+15 : 85-3x$$

$$\therefore 2x(85-3x)=(x+5)(3x+15)$$

$$\text{or } 170x-6x^2=3x^2+30x+75 \therefore 9x^2-140x=-75$$

$$\therefore x^2-\frac{140}{9}+\left(\frac{70}{9}\right)^2=\left(\frac{70}{9}\right)^2-\frac{75}{9}=\frac{4225}{81}$$

$$\therefore\ x-\frac{70}{9}=\pm\frac{65}{9}\ \therefore\ x=15 \text{ or } \frac{5}{9}.$$

Rejecting the second value, we have

$$40x=£600 \text{ A's stock};\ 20x+100=£400, \text{ B's stock}$$
$$3x+15=£60, \text{ A's gain};\ 85-3x=£40, \text{ B's gain}.$$

(29.) Suppose B went $2x$ miles a day, then A went $2x+8$, and, by the question, they met in x days. Consequently, the part of the 320 miles travelled by A was $2x^2+8x$ miles, and the part travelled by B was $2x^2$ miles; therefore, adding these parts,

$$4x^2+8x=320\ \therefore\ 4x^2+8x+4=324;$$
$$\text{that is, } (2x+2)^2=324\ \therefore\ 2x+2=\pm 18\ \therefore\ x=8$$

$\therefore$ A went $2x+8=24$ miles a day; and B, $2x=16$ miles a day; A's part of the distance was $2x^2+8x=192$ miles, and B's $2x^2=128$.

(30.) Let x and y be the depths in feet: then the side of the square base of the former is y, and that of the latter x. The capacities of the vats are therefore xy^2, yx^2; and by the question,

$$xy^2=yx^2+20 \text{ and } 4xy^2=5yx^2\ \therefore\ x=\frac{4}{5}y$$

$$\therefore\ \frac{4}{5}y^3=\frac{16}{25}y^3+20\ \therefore\ \frac{4}{25}y^3=20\ \therefore\ y^3=5^3\ \therefore\ y=5\ \therefore\ x=\frac{4}{5}y=4$$

Hence the depths are 5 feet and 4 feet.

(31.) Let the sum of money be represented by n^3x, and for a write n^3b; then, after A's deduction, there remains

$$n^3x-n^2(x+b)=n^2(n-1)x-n^2b.$$

After B's, $n^2\{(n-1)x-b\}-n\{(n-1)x-b\}-n^2b=$
$n(n-1)\{(n-1)x-b\}-n^2b$; and after C's,

$$n(n-1)\{(n-1)x-b\}-n^2b-(n-1)\{(n-1)x-b\}$$
$$+nb-n^2b=0; \text{ that is,}$$

$$(n-1)^3x-(n-1)^2b-2n^2b+nb=0\ \therefore\ x=\frac{3n^2-3n+1}{(n-1)^3}b.$$

Hence, putting $\frac{a}{n^3}$ for b, we have for the sum $n^3x=\frac{3n^2-3n+1}{(n-1)^3}a.$

(32.) The first remainder is evidently $a\left(1-\frac{1}{m}\right)$:

the second is $a\left(1-\frac{1}{m}\right)\left(1-\frac{1}{n}\right)$,

the fourth is $a\left(1-\frac{1}{m}\right)^2\left(1-\frac{1}{n}\right)^2$,

the sixth is $a\left(1-\frac{1}{m}\right)^3\left(1-\frac{1}{n}\right)^3$.

And generally the $2p$th is $a\left(1-\frac{1}{m}\right)^p\left(1-\frac{1}{n}\right)^p$.

This is the remainder left after $2p$ days' journey, so that a diminished by it is the distance gone over in $2p$ days; that is, the distance gone over is

$$a\left\{1-\left(1-\frac{1}{m}\right)^p\left(1-\frac{1}{n}\right)^p\right\}.$$

INEQUALITIES (Page 103).

(5.) Taking the square of each, we have to consider

$17+2\sqrt{70}$ and $21+2\sqrt{57}$,

or $2\sqrt{70}$ and $4+2\sqrt{57}$, or $\sqrt{70}$ and $2+\sqrt{57}$,

or squaring, 70 and $61+4\sqrt{57}$, or 9 and $4\sqrt{57}$.

The latter quantity is evidently the greater, so that

$$\sqrt{3}+\sqrt{19}>\sqrt{7}+\sqrt{10}.$$

(6.) From the first inequality, $4x-7<2x+3$, $2x<10$ $\therefore$ $x<5$.

From the second, $3x+1>13-x$, $4x>12$ $\therefore$ $x>3$.

And the only integer less than 5 and greater than 3 is 4.

(7.) The expressions $\frac{a+b}{2}$, $\frac{2ab}{a+b}$ in a common denominator are $\frac{a^2+2ab+b^2}{2(a+b)}$, $\frac{4ab}{2(a+b)}$: hence we have to compare a^2+b^2 and $2ab$, or $(a-b)^2$ and 0. The former is always the greater except when $a=b$.

Again, the expressions a^5+b^5, a^4b+ab^4 are unequal only when a and b are. Let $a>b$, that is, let $a=b+x$; then the expressions to be compared are

$$(b+x)^5+b^5, \text{ and } (b+x)^4b+(b+x)b^4,$$
$$\text{or } 2b^5+5b^4x+10b^3x^2+10b^2x^3+5bx^4+x^5,$$
$$\text{and } b^5+4b^4x+6b^3x^2+4b^2x^3+bx^4+b^5+b^4x,$$
$$\text{or } 4b^3x^2+6b^2x^3+4bx^4+x^5 \text{ and } 0,$$

of which the former is the greater, a and b being regarded as positive.

Otherwise. $a^5+b^5-(a^4b+ab^4)=a^4(a-b)-b^4(a-b)=(a^4-b^4)(a-b)$. Now whether a be greater or less than b, these two factors must have the same sign: therefore, the first member of the equation must be positive $\therefore a^5+b^5>a^4b+ab^4$.

(8.) Since $3(n^2-n+1)-(n^2+n+1)=2n^2-4n+2=2(n-1)^2$ is a positive quantity, except in the single case $n=1$

$$\therefore 3(n^2-n+1)>n^2+n+1 \therefore \frac{n^2-n+1}{n^2+n+1}>\frac{1}{3}.$$

Again: $3(n^2+n+1)-n^2-n+1=2n^2+4n+2=2(n+1)^2$ is a positive quantity, except in the single case $n=-1$

$$\therefore 3(n^2+n+1)>n^2-n+1 \therefore \frac{n^2-n+1}{n^2+n+1}<3.$$

The extreme limits of the real values of the proposed fraction are therefore 3 and $\frac{1}{3}$: which extreme values are furnished by $n=-1$, and $n=1$: for all other values of n the fraction lies between the limits 3 and $\frac{1}{3}$.

Proportion, Variation, &c. (Page 114).

(1.) $\frac{3}{7}, \frac{4}{9}=\frac{27}{63}, \frac{28}{63}$ $\therefore$ the greater ratio is $4:9$.

(2.) $\frac{2}{9}\times\frac{12}{5}=\frac{8}{15}$ $\therefore$ the compound ratio is $8:15$.

(3.) $\frac{m}{6x^2}\times\frac{3y^2}{n}\times\frac{x^2}{2y^2}=\frac{m}{4n}$ $\therefore$ the compound ratio is $m:4n$.

(4.) The expressions to be compared are $x-y$ and $(x^{\frac{1}{2}}-y^{\frac{1}{2}})^2$, or $x^{\frac{1}{2}}+y^{\frac{1}{2}}$ and $x^{\frac{1}{2}}-y^{\frac{1}{2}}$, the former of which exceeds the latter by $2y^{\frac{1}{2}}$.

(5.) Since A varies as B $\therefore$ A$=m$B. Similarly B$=n$C $\therefore$ A$=mn$C; that is, A varies as C.

(6.) If $a:b::c:d$, then $\frac{a}{b}=\frac{c}{d}$ $\therefore \frac{ma}{b}\pm n=\frac{mc}{d}\pm n$.

$$\text{Also, } \frac{pa}{b}\pm q=\frac{pc}{d}\pm q.$$

Hence, dividing the former equation by this, we have

$$\frac{ma\pm nb}{pa\pm qb}=\frac{mc\pm nd}{pc\pm qd}. \quad \text{Consequently,}$$

$$ma\pm nb:pa\pm qb::mc\pm nd:pc\pm qd.$$

(7.) These are the same as the equations

$$x=y+12 \text{ and } 5\sqrt{xy}=2(x+y). \dots (A).$$

Squaring the second, $25xy=4(x+y)^2$.

Also from the first, $4 \,.\, 12^2=4(x-y)^2$.

Subtracting, $25xy-4 \,.\, 12^2=16xy$ $\therefore 9xy=4 \,.\, 12^2$ $\therefore xy=64$

$\therefore$ (A), $20=x+y$, and $12=x-y$ $\therefore x=16$, $y=4$.

(8.) Here we have to compare

$$\sqrt{(a^2-b^2)}+\sqrt{\{a^2-(a-b)^2\}} \text{ and } a. \dots (A),$$

$$\text{or } \sqrt{\{a^2-(a-b)^2\}} \text{ and } a-\sqrt{(a^2-b^2)};$$

$$\text{or squaring, } 2ab-b^2 \text{ and } 2a^2-2a\sqrt{(a^2-b^2)}-b^2,$$

$$\text{or } b \text{ and } a-\sqrt{(a^2-b^2)},$$

$$\text{or } (a^2-b^2) \text{ and } (a-b)^2, \text{ or } 2ab \text{ and } 2b^2,$$

the former of which is the greater, because $a>b$: hence the first of (A) is greater than the second.

(9.) Put $y=mx$, then by the conditions $10=2m$

$\therefore m=5$ $\therefore y=5x$.

(10.) Let $3x$ be one number, then $2x$ is the other; and by the question, $5x\times 6x^2=12(9x^2-4x^2)$; that is, $30x^3=60x^2$

$\therefore x=2$ $\therefore 3x=6$ and $2x=4$.

(11.) Put $y^2=m(a^2-x^2)$ $\therefore$ when $x=0$, $b^2=ma^2$

$$\therefore m=\frac{b^2}{a^2} \therefore y^2=\frac{b^2}{a^2}(a^2-x^2).$$

(12.) These are the same as the equations

$$b^3x=a^3y, \text{ and } b\sqrt[3]{(c+x)}=a\sqrt[3]{(d+y)}.$$

Cubing the second, $b^3(c+x)=a^3(d+y)$.

Subtracting the first, $b^3c=a^3d$.

Multiplying this by the first reversed, $a^3b^3cy=a^3b^3dx \therefore dx=cy$.

(13.) Let x be the first number and y the common ratio, then the three numbers are x, xy, and xy^2; and by the question,

$$x+xy+xy^2=52\ldots.(1).$$

Also $x+xy^2 : xy :: 10 : 3 \therefore 3(x+xy^2)=10xy\ldots.(2)$.

From the first, $x+xy^2=52-xy$.

Substituting in the second, $3(52-xy)=10xy$

$\therefore xy=12$, the mean or second number

$$\therefore (1),\ x+xy^2=40 \therefore 1+y^2=\frac{40}{x};\ \text{but } x=\frac{12}{y} \therefore 1+y^2=\frac{10y}{3}$$

$$\therefore y^2-\frac{10}{3}y=-1 \therefore y^2-\frac{10}{3}y+\frac{25}{9}=\frac{16}{9} \therefore y=3,\ \text{or } \tfrac{1}{3}$$

$$\therefore x=\frac{12}{y}=4,\ \text{or } 36:\ \text{hence the numbers are } 4,\ 12,\ 36.$$

Arithmetical Progression (Page 121).

(1.) $S=1+2+3+4+\ldots\ldots+n=\frac{1}{2}n(n+1)$.

(2.) $3(3+5+7+\text{to ten terms})$. Here $a=3$, $d=2$, $n=10$;

$\therefore S=3n\{a+\frac{1}{2}(n-1)d\}=30\{3+9\}=360$.

(3.) Here $a=\frac{1}{2}$, $d=\frac{1}{3}-\frac{1}{2}=-\frac{1}{6}$, $n=20$;

$$\therefore l=a+(n-1)d=\frac{1}{2}-\frac{19}{6}=-\frac{16}{6}=-2\frac{2}{3}$$

$$S=\frac{n}{2}(a+l)=10\left(\frac{1}{2}-2\frac{2}{3}\right)=-\frac{130}{6}=-21\frac{2}{3}.$$

(4.) Here $a=1$, $d=7$, $n=100$;

$\therefore S=n\{a+\frac{1}{2}(n-1)d\}=100\{1+346\frac{1}{2}\}=34750$.

(5.) Here $a=\frac{1}{3}$, $d=\frac{1}{4}-\frac{1}{3}=-\frac{1}{12}$, $n=12$;

$$\therefore S=n\{a+\tfrac{1}{2}(n-1)d\}=12\left\{\frac{1}{3}-\frac{11}{24}\right\}=4-5\tfrac{1}{2}=-1\tfrac{1}{2}.$$

(6.) Here $a=7$, $d=5\frac{1}{2}-7=-1\frac{1}{2}$, $n=9$;
$\therefore l=a+(n-1)d=7-12=-5$.

(7.) Here $a=\frac{7}{12}$, $d=\frac{2}{3}-\frac{7}{12}=\frac{1}{12}$, $n=24$;

$$\therefore l=a+(n-1)d=\frac{7}{12}+\frac{23}{12}=\frac{30}{12}=2\frac{1}{2}$$

$$S=\frac{n}{2}(a+l)=12\left(\frac{7}{12}+\frac{30}{12}\right)=37.$$

(8.) Here we have the equation
$l=a+(n-1)d$; that is, $17=3+28d$, to find d;

$\therefore d=\frac{14}{28}=\frac{1}{2}$ $\therefore$ the series is 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, &c.

(9.) Here $a=1$, $d=8$, $n=100$;
$\therefore l=a+(n+1)d=1+792=793$.

(10.) Here $a=\frac{1}{2}$, $d=\frac{3}{4}-\frac{1}{2}=\frac{1}{4}$, $n=10$;

$$\therefore S=n\left\{a+\frac{1}{2}(n-1)d\right\}=10\left\{\frac{1}{2}+\frac{9}{8}\right\}=16\tfrac{1}{4}.$$

(11.) Here $a=198$, $d=-5$, $n=40$;

$$\therefore S=n\left\{a+\frac{1}{2}(n-1)d\right\}=40\left\{198-\frac{195}{2}\right\}=4020.$$

(12.) Here $a=\frac{1}{2}$, $d=\frac{1}{6}-\frac{1}{2}=-\frac{1}{3}$, $n=20$;

$$\therefore l=a+(n-1)d=\frac{1}{2}-\frac{19}{3}=\frac{35}{6}=5\frac{5}{6}.$$

(13.) Here $a=\frac{1}{2}$, $d=-\frac{2}{3}-\frac{1}{2}=-\frac{7}{6}$, $n=13$;
$\therefore S=n\{a+\frac{1}{2}(n-1)d\}=13\{\frac{1}{2}-7\}=-84\frac{1}{2}$.

(14.) Here $a=\frac{5}{7}$, $d=\frac{4}{7}$, $n=n$;

$$\therefore S=n\left\{a+\frac{1}{2}(n-1)d\right\}=n\left\{\frac{5}{7}+\frac{2(n-1)}{7}\right\}=\frac{n}{7}(2n+3).$$

(15.) Here $a=19$ and $n=5$, also $l=35$, to find d.

$l=a+(n-1)d \therefore 35=19+4d \therefore d=4$.

Hence the three means between 19 and 35 are 23, 27, 31.

Again: $a=2\frac{2}{3}$, $n=7$, and $l=\frac{2}{3}$, to find d

$l=a+(n-1)d \therefore 2\frac{2}{3}=2\frac{2}{3}+6d \therefore d=-\frac{1}{3}$.

Hence the five means between $2\frac{2}{3}$ and $\frac{2}{3}$ are 2, $1\frac{2}{3}$, $1\frac{1}{3}$, 1.

(16.) Here $d=5$, $n=15$ and $S=600$, to find a.

$S=n\{a+\frac{1}{2}(n-1)d\} \therefore 600=15\{a+35\}$

$\therefore 40=a+35 \because a=5$.

(17.) $S=n\{a+\frac{1}{2}(n-1)d\} \therefore 40=n\{7+n-1\}=n^2+6n$;

$\therefore n^2+6n+9=49 \therefore n+3=\pm 7 \therefore n=4$, or -10.

(18.) Here $a=3$, $n=10$, $S=165$, to find d.

$$S=n\{a+\tfrac{1}{2}(n-1)d\} \therefore 165=10\left\{3+\frac{9}{2}d\right\}=30+45d$$

$$\therefore d=\frac{135}{45}=3 \therefore \text{the prog. is } 3, 6, 9, 12, \text{\&c.}$$

(19.) Here $a=2\frac{1}{2}$, $n=6$, $l=6\frac{1}{4}$, to find d.

$$l=a+(n-1)d \therefore 6\tfrac{1}{4}=2\tfrac{1}{2}+5d \therefore d=\frac{15}{20}=\frac{3}{4}.$$

Hence the four means, are $3\frac{1}{4}$, 4, $4\frac{3}{4}$, $5\frac{1}{2}$.

(20.) Here $n=9$, $l=-\frac{1}{3}$, and $S=0$, to find a and d.

$$S=\frac{n}{2}(a+l) \therefore 0=\frac{9}{2}\left(a-\frac{1}{3}\right) \therefore a=\frac{1}{3}.$$

$$\text{Also } l=a+(n-1)d \therefore -\frac{1}{3}=\frac{1}{3}+8d \therefore d=-\frac{1}{12}.$$

$$\text{Hence the progression is } \frac{1}{3}+\frac{1}{4}+\frac{1}{6}+ \text{\&c.}$$

(21.) Here $a=-1$, $n=5$, $l=15$, to find d.

$l=a+(n-1)d \therefore 15=-1+4d \therefore d=4$.

Hence the progression is -1, 3, 7, 11, 15, the three means being inserted.

(22). Here $a=1\frac{1}{4}$, $d=-\frac{1}{8}$, and $S=6\frac{7}{8}$, to find n.

$$S=n\{a+\tfrac{1}{2}(n-1)d\}$$

$$\therefore 6\tfrac{7}{8}=n\left\{1\tfrac{1}{4}-\frac{1}{16}(n-1)\right\}=\frac{21}{16}n-\frac{1}{16}n^2$$

$$\therefore n^2-21n=-110 \therefore n^2-21n+\left(\frac{21}{2}\right)^2=\frac{1}{2}$$

$$\therefore n-\frac{21}{2}=\pm\frac{1}{2} \therefore n=10.$$

(23.) Two terms are interposed between 7 and 16; namely, the fourth and fifth: hence, by inserting two means between these extremes, we shall find d, from having given

$$a=7,\ n=4,\ l=16.$$

$$l=a+(n-1)d \therefore 16=7+3d \therefore d=3.$$

Hence the series is 1, 4, 7, 10, 13, 16, &c.

(24.) In this example, d is evidently $\frac{30}{x+1}$, and therefore the $(x-1)$th mean—that is, the xth term—is $1+(x-1)d=1+\frac{30(x-1)}{x+1}$, and the seventh mean, or eighth term, is $1+7d=1+\frac{210}{x+1}$. And by the condition,

$$1+\frac{210}{x+1}=\frac{5}{9}\left(1+\frac{30x+30}{x+1}\right)$$

$$\therefore 9(x+1)+1890=5(x+1)+150x-150$$

$$\therefore 4(x+1)-150x=-2040 \therefore 146x=2044 \therefore x=14.$$

Hence the number of means is fourteen.

(25.) Let x be the first term, and y the common difference, then the progression is x, $x+y$, $x+2y$, and by the question,

$$3x+3y=10 \therefore x+y=3\tfrac{1}{3},\text{ the mean term}$$

$$\text{also } (x+y)(x+2y)=33\tfrac{1}{3} \therefore 3\tfrac{1}{3}(x+2y)=33\tfrac{1}{3}$$

$$\therefore x+2y=33\tfrac{1}{3}\div 3\tfrac{1}{3}=10,\text{ the third term,}$$

$$\text{and } 2(x+y)-(x+2y)=6\tfrac{2}{3}-10=-3\tfrac{1}{3},\text{ the first term.}$$

(26.) Let the three numbers be $x-y$, x, $x+y$; then we are to have $3x=15 \therefore x=5$; also $(x-y)^2+x^2+(x+y)^2=93$.

The second equation is $3x^2+2y^2=93$ $\therefore$ $75+2y^2=93$ $\therefore$ $y^2=9$ $\therefore$ $y=3$, and since $x=5$ the numbers are 2, 5, and 8.

(27.) In this example, $a=\dfrac{x^2-1}{x}=x-\dfrac{1}{x}$, $d=\dfrac{1}{x}$, $n=n$.

$$l=a+(n-1)d \therefore l=x-\frac{1}{x}+(n-1)\frac{1}{x}=x+(n-2)\frac{1}{x}$$

$$S=\frac{n}{2}(a+l)=\frac{n}{2}\left\{\left(x-\frac{1}{x}\right)+x+(n-2)\frac{1}{x}\right\}=nx+\frac{n}{2}(n-3)\frac{1}{x}.$$

(28.) Let the three numbers be $x-y$, x, $x+y$; then we are to have $3x=24$ $\therefore$ $x=8$; also $x(x^2-y^2)=480$.

Substituting the first in the second, $8(64-y^2)=480$

$\therefore$ $64-y^2=60$ $\therefore$ $y^2=4$ $\therefore$ $y=2$

$\therefore$ the numbers are 6, 8, 10.

(29.) Here $a=n^2-n+1$, $d=2$, $n=n$.

$S=n\{a+\frac{1}{2}(n-1)d\}$ $\therefore$ $S=n\{n^2-n+1+n-1\}=n^3$.

(30.) Let the numbers be $x-3y$, $x-y$, $x+y$, $x+3y$; then we are to have $x^2-9y^2=27$, and $x^2-y^2=35$.

Subtracting the first from the second, $8y^2=8$ $\therefore$ $y=1$

$\therefore$ $x^2=36$ $\therefore$ $x=6$: hence the numbers are 3, 5, 7, 9.

(31.) Here $l=a+(n-1)d=m$, and $l_1=a+(m-1)d=n$. Hence, to determine a and d, we have the two equations

$$a+(n-1)d=m$$
$$a+(m-1)d=n.$$

Subtracting, $(n-m)d=m-n$ $\therefore$ $d=\dfrac{m-n}{n-m}=-1$

$\therefore$ $a=m-(n-1)d=m+n-1$.

Put now n_1 for the required number of terms, then

$$S=n_1\{a+\tfrac{1}{2}(n_1-1)d\}$$

$$\therefore \tfrac{1}{2}(m+n)(m+n-1)=n_1\{m+n-1-\tfrac{1}{2}(n_1-1)\}$$

$$\therefore n_1^2-(2m+2n-1)n_1+(m+n)(m+n-1)=0.$$

A mere inspection of this equation shows that the sum of the factors of the last term, taken with opposite signs, forms

the coefficient of the second term; therefore (Note, p. 60), the roots are $n_1 = m+n$, or $n_1 = m+n-1$.
For the last term we have, since $d=-1$,

$$l = a+(n_1-1)d = m+n-1-m-n+1=0,$$
$$\text{or } = m+n-1-m-n+1+1=1.$$

(32.) Here $a=4$, $n=60$, $l=88$.

$$S=\frac{n}{2}(a+l)=30(4+88)=2760 \text{ feet.}$$

(33.) To find d in the last example, we have

$$l=a+(n-1)d \therefore 88=4+59d \therefore d=\frac{84}{59}=1\frac{25}{59} \text{ feet.}$$

To find the time occupied in travelling 5280 feet, we have to determine n, the number of seconds, from the equation

$$S=n\{a+\tfrac{1}{2}(n=1)d\} \therefore 5280=n\left\{4+\frac{42}{59}(n-1)\right\}$$

$$\therefore 5280\times 59=236n+42n^2-42n$$

$$\therefore 42n^2+194n=5280\times 59$$

$$\therefore \quad n^2+\frac{97}{21}n=\frac{2640\times 59}{21}$$

$$\therefore n^2+\frac{97}{21}n+\left(\frac{97}{42}\right)^2=\frac{13093249}{42^2}$$

$$\therefore n=\frac{\sqrt{13093249}-97}{42}=\frac{3521\cdot46}{42} \text{ nearly}$$

$$=83\cdot86 \text{ or } 83\tfrac{2}{2}\tfrac{1}{5} \text{ nearly.}$$

(34.) Here $l=4a \therefore S=\frac{n}{2}(5a)$; but $S=\frac{3a^2}{4}$, and $n=\frac{1}{2}d$;

$$\therefore l=4a=a+(n-1)2n,\ S=\frac{3a^2}{4}=\frac{n}{2}(5a) \therefore n=\frac{3a}{10}$$

$$\therefore 4a=a+\left(\frac{3a}{10}-1\right)\frac{3a}{5}$$

$$\therefore 1=\left(\frac{3a}{10}-1\right)\frac{1}{5} \therefore 60=3a \therefore a=20$$

$$\therefore n=\frac{3a}{10}=6 \therefore d=2n=12$$

$\therefore$ the series is 20, 32, 44, 56, 68, 80.

(35.) Suppose that B comes up to A in x days after B starts, then B will have travelled x days, and have gone over $9ax$ miles. This, therefore, will be the distance A has gone over in $x+4$ days.

$$S=n\{a+\tfrac{1}{2}(n-1)d\} \therefore 9ax=(x+4)\{a+\tfrac{1}{2}(x+3)a\};$$

$$\text{that is, } 9ax=ax+4a+\tfrac{1}{2}ax^2+\frac{7}{2}ax+6a$$

$$\therefore \frac{9}{2}ax=\tfrac{1}{2}ax^2+10a \therefore x^2-9x=-20$$

$$\therefore x^2-9x+\frac{81}{4}=\frac{1}{4} \therefore x-\frac{9}{2}=\pm\frac{1}{2} \therefore x=4, \text{ or } 5.$$

NOTE.—It thus appears that B will come up to A in 4 days; will accompany him, and then be left behind by A at the close of the next day's journey: thus,

A's journeys,	a, $2a$, $3a$, $4a$, $5a$, $6a$, $7a$, $8a$, $9a$
B's ,,	$9a$, $9a$, $9a$, $9a$, $9a$.

In this latter way, the example may be very easily solved: when B starts, A will be $a+2a+3a+4a=10a$ miles in advance. Four days after, A will have increased his distance by $5a+6a+7a+8a=26a$ miles, making $36a$ miles altogether; and B will have gone $9a\times4=36a$, the same distance. Then each goes $9a$ miles on the fifth day, reaching the end of that day's journey together; after which A takes the lead, and B never again comes up to him, for the distance between them increases by a miles every day.

(36.) Suppose there were $2x$ persons, and that the youngest paid y pounds; then, since the younger half, consisting of x persons, paid $22x$ pounds altogether, and that the whole $2x$ persons paid 345 pounds, we have

$$S= n\{a+\tfrac{1}{2}(n-1)d\}$$

$$22x= x\{y+\tfrac{1}{2}(x-1)5\}\ldots.(A)$$

$$345=2x\{y+\tfrac{1}{2}(2x-1)5\}\ldots.(B).$$

Subtracting twice (A) from (B),

$$345-44x=2x\left\{\frac{5}{2}x\right\}=5x^2$$

$$\therefore\ 5x^2+44x=345\ \therefore\ x^2+\frac{44}{5}x=69$$

$$\therefore\ x^2+\frac{44}{5}x+\left(\frac{22}{5}\right)^2=\frac{2209}{25}\ \therefore\ x=\frac{-22\pm 47}{5}=5, \text{ or } -\frac{69}{5}.$$

Hence $2x=10$, the number of persons.

(37.) The sum of $n+1$ terms is

$$S=(n+1)\{a+\tfrac{1}{2}nd\}=(n+1)(n+1\tfrac{1}{3})$$

$$\therefore\ a+\tfrac{1}{2}nd=n+1\tfrac{1}{3}\ \therefore\ a+nd=2n+2\tfrac{2}{3}-a\ldots\ (A).$$

The nth term is $a+(n-1)d$; that is, substituting (A),

nth term $=2n+2\frac{2}{3}-(a+d)=2n+2\frac{2}{3}-$ second term.

But the second term is given by putting 1 for n in the expression for the sum of $n+1$ terms, and then subtracting the first term, or what that expression gives for $n=0$; namely, $1\frac{1}{3}$

$$\therefore\ n\text{th term } =2n+2\tfrac{2}{3}-3\tfrac{1}{3}=2(n-\tfrac{1}{3}).$$

NOTE.—The example may be solved otherwise, thus: putting $n=0$, the first term is $1\frac{1}{3}$: putting $n=1$, the sum of two terms is $4\frac{2}{3}$: hence we have $a=1\frac{1}{3}$, $d=4\frac{2}{3}-2\frac{2}{3}=2$, to find the expression for the nth term, which is

$$1\tfrac{1}{3}+(n-1)2=2(n-\tfrac{1}{3}).$$

(38.) Putting 1 for n, the first term is $p+q$; putting 2 for n, the sum of the first and second terms is $2p+4q$. Subtracting twice the first from this, we have $d=2q$: hence we have

$$a=p+q,\ d=2q,\ n=m$$

$$l=a+(n-1)d=p+q+(m-1)2q=p+(2m-1)q.$$

(39.) The proposed expression is the sum of n terms of an arithmetical progression: hence, putting $n=1$, the first term is $\frac{5}{7}$: putting $n=2$, the sum of the first and second terms is $\frac{12}{7}$; therefore, subtracting twice the former from this, we have $d=\frac{2}{7}$. Consequently, the series is

$$\frac{5}{7},\ 1,\ 1\frac{2}{7},\ 1\frac{4}{7},\ \text{\&c., the parts required.}$$

(40.) Let x be the first term, and y the common difference; then, putting $l=a$, b, and c, successively, we have

$$\left.\begin{array}{l} a=x+(p-1)y \\ b=x+(q-1)y \\ c=x+(r-1)y \end{array}\right\} \therefore \left.\begin{array}{l} a-b=(p-q)y \\ \\ b-c=(q-r)y \end{array}\right\} \therefore \frac{a-b}{b-c}=\frac{p-q}{q-r}$$

$$\therefore (q-r)a-(q-r)b=(p-q)b-(p-q)c$$

$$\therefore (q-r)a+(r-p)b+(p-q)c=0.$$

Geometrical Progression (Page 133).

(1.) Here $a=3$, $r=2$, $n=6$;

$$\therefore S=a\frac{r^n-1}{r-1}=3\frac{2^6-1}{1}=3\times 63=189.$$

(2.) Here $a=5$, $r=4$, $n=5$;

$$\therefore S=a\frac{r^n-1}{r-1}=5\frac{4^5-1}{3}=5\times 341=1705.$$

(3.) Here $a=\frac{3}{2}$, $r=\frac{2}{3}$, $n=6$;

$$\therefore S=a\frac{r^n-1}{r-1}=\frac{3}{2}\frac{\left(\frac{2}{3}\right)^6-1}{-\frac{1}{3}}=\frac{1995}{729}\times\frac{3}{2}=4\frac{17}{162}.$$

(4.) Here $a=9$, $r=-\frac{2}{3}$, $n=8$;

$$\therefore l=ar^{n-1}=9\left(-\frac{2}{3}\right)^7=-\frac{128}{243}.$$

(5.) Here $a=3$, $r=\frac{1}{6}$, $n=6$;

$$\therefore l=ar^{n-1}=3\left(\frac{1}{6}\right)^5=\frac{1}{2592}.$$

(6.) Here $a=1$, $r=-2$, $n=10$;

$$\therefore S=a\frac{r^n-1}{r-1}=\frac{2^{10}-1}{-3}=-341.$$

(7.) Here $a=21$, $r=-\frac{1}{7}$, $n=5$;

$$\therefore\ l=ar^{n-1}=21\left(-\frac{1}{7}\right)^4=\frac{3}{343}.$$

$$\text{And } S=\frac{rl-a}{r-1}=-\left(\frac{3}{7\times 343}+21\right)\div-\frac{8}{7}$$

$$=\frac{50424}{8\times 343}=\frac{6303}{343}=18\frac{129}{343}.$$

(8.) Here $a=-\frac{1}{2}$, $r=-\frac{2}{3}$, $n=5$;

$$\therefore\ l=ar^{n-1}=-\frac{1}{2}\left(\frac{2}{3}\right)^4=-\frac{8}{81}.$$

(9.) Here $a=\frac{4}{5}$, $r=\frac{5}{2}$, $n=n$; $\therefore\ l=ar^{n-1}=\frac{4}{5}\left(\frac{5}{2}\right)^{n-1}=\frac{5^{n-2}}{2^{n-3}}.$

$$S=\frac{rl-a}{r-1}=\left(\frac{5}{2}\cdot\frac{5^{n-2}}{2^{n-3}}-\frac{4}{5}\right)\div\frac{3}{2}=\frac{5^n-2^n}{5\,.\,2^{n-2}}\times\frac{2}{3}$$

$$=\frac{1}{15}\cdot\frac{5^n-2^n}{2^{n-3}}.$$

(10.) Here $a=\frac{3}{5}$, $r=\frac{5}{6}$, $n=8$ $\therefore\ S=a\frac{r^n-1}{r-1}=\frac{3}{5}\cdot\dfrac{\left(\frac{5}{6}\right)^8-1}{-\frac{1}{6}}$

$$=-3\left\{\left(\frac{5}{6}\right)^7-\frac{6}{5}\right\}=\frac{6^8-5^8}{10\,.\,6^6}$$

$$\therefore\ S=\frac{1288991}{466560}.$$

(11.) Here $a=\frac{1}{4}$, $r=-\frac{1}{4}$ $\therefore\ S=\frac{a}{1-r}=\dfrac{\frac{1}{4}}{\frac{5}{4}}=\frac{1}{5}.$

(12.) Here $a=1$, $r=-\frac{1}{2}$ $\therefore\ S=\frac{a}{1-r}=\dfrac{1}{\frac{3}{2}}=\frac{2}{3}.$

(13.) Here $a=2$, $r=-\frac{2}{3}$ $\therefore$ $S=\frac{a}{1-r}=\frac{2}{\frac{5}{3}}=1\frac{1}{5}$.

(14.) Here $a=-3\frac{1}{5}$, $r=-\frac{1}{2}$ $\therefore$ $S=\frac{a}{1-r}=-\frac{3\frac{1}{5}}{\frac{3}{2}}=-2\frac{2}{15}$.

(15.) Here $a=\frac{1}{2}$, and $ar^4=128=\frac{1}{2}r^4$ $\therefore$ $r=\sqrt[4]{256}=\pm 4$.

Hence the means are $ar=\pm 2$, $ar^2=8$, $ar^3=\pm 32$.

(16.) Here $a=5$, and $ar^3=1080=5r^3$ $\therefore$ $r=\sqrt[3]{216}=6$.

Hence the two means are $ar=30$, $ar^2=180$.

Again: $a=9$, and $ar^4=\frac{1}{9}=9r^4$ $\therefore$ $r=\sqrt[4]{\frac{1}{9^2}}=\pm\frac{1}{3}$.

Hence the three means are $ar=\pm 3$, $ar^2=1$, $ar^3=\pm\frac{1}{3}$.

(17.) The conditions are $S=\frac{a}{1-r}=2$, and $a+ar=1\frac{1}{2}$.

From the first, $a=2-2r$ $\therefore$ $2-2r+2r-2r^2=1\frac{1}{2}$

$\therefore$ $2r^2=\frac{1}{2}$ $\therefore$ $r=\pm\frac{1}{2}$ $\therefore$ $a=2-2r=1$ or 3.

Hence the series is either

$$1+\frac{1}{2}+\frac{1}{4}+\text{\&c., or } 3-\frac{3}{2}+\frac{3}{4}-\text{\&c.}$$

NOTE.—Since the sum of either series is 2, it follows that

$$1+\frac{1}{2}+\frac{1}{4}+\text{ \&c., to infinity }=3\left(1-\frac{1}{2}+\frac{1}{4}-\text{\&c. to infinity.}\right)$$

And, generally, an infinite decreasing geometrical series, whose terms are all positive, is equal to the same series, with signs alternately positive and negative, multiplied by $\frac{1+r}{1-r}$; for this fraction evidently expresses the ratio of the two sums. In the present example, r, in the series $1+\frac{1}{2}+\frac{1}{4}$, &c., is $\frac{1}{2}$; and

therefore $\frac{1+r}{1-r}=\frac{3}{2}\times\frac{2}{1}=3$, so that the series is three times the series $1-\frac{1}{2}+\frac{1}{4}-$ &c.

(18.) Let x be the first number, and y the common ratio; then by the question,

$$x^3y^3=64, \text{ and } x^3+x^3y^3+x^3y^6=584.$$

From the first of these,

$$y^3=\frac{64}{x^3} \therefore y^6=\frac{4096}{x^6}, \text{ and the second is}$$

$$x^3+64+\frac{4096}{x^3}=584 \therefore x^6+64x^3+4096=584x^3$$

$$\therefore x^6-520x^3=-4096 \therefore x^6-520x^3+260^2=63504$$

$$\therefore x^3-260=\pm 252 \therefore x^3=8, \text{ or } 512$$

$$\therefore x=2, \text{ or } 8 \therefore y=\frac{4}{x}=2, \text{ or } \frac{1}{2};$$

hence the numbers are 2, 4, 8.

(19.) Let Σ represent the sum to infinity; then

$$S=a\frac{r^n-1}{r-1}, \text{ and } \Sigma=\frac{a}{1-r}$$

$$\therefore \frac{a}{1-r}=2a\frac{r^n-1}{r-1} \because -1=2(r^n-1) \therefore r=\left(\frac{1}{2}\right)^{\frac{1}{n}}.$$

(20.) Let the numbers be x, y; then the arithmetic mean is $\frac{x+y}{2}$, and the geometric mean is $\sqrt{xy}$; and by the conditions,

$$\left.\begin{array}{l}\frac{1}{2}(x+y)+\sqrt{xy}=13\frac{1}{2}\\ \frac{1}{2}(x+y)-\sqrt{xy}=\ \ 1\frac{1}{2}\end{array}\right\} \therefore x+y=15,\ \sqrt{xy}=6$$

$$\therefore (x+y)^2-4xy=(x-y)^2=225-144=81 \therefore x-y=\pm 9$$

$$\therefore x=3,\ y=12;\ \text{or } x=12,\ y=3.$$

Hence the numbers are 3 and 12.

(21.) Suppose the leveret runs x yards before being caught; then the greyhound runs $100x \therefore 100x=100+x \therefore 99x=100$ $\therefore x=1\frac{1}{99} \therefore 100+x=101\frac{1}{99}$, yards run by the greyhound.

(22.) Let P represent the population at any period, and r

the rate of annual increase; then $P(1+r)$ is the state P_1 at the end of 1 year; $P_1(1+r) = P(1+r)^2$ is the state P_2 at the end of 2 years; and so on. Hence, generally, the state P_n at the end of n years is $P(1+r)^n$. In the present example, $P=10000$, and $n=4$; and the condition is that

$$10000(1+r)^4=14641 \therefore 1+r=\frac{\sqrt[4]{14641}}{10}=\frac{11}{10} \therefore r=\frac{1}{10},$$

the rate of annual increase.

(23.) The first two conditions are

$$m=ar^{p+q-1}, \text{ and } n=ar^{p-q-1}$$

$$\therefore mn=a^2r^{2p-2} \therefore \sqrt{mn}=ar^{p-1}, \text{ the } p\text{th term.}$$

Also $\frac{m}{n}=r^{2q}$, and $\frac{m}{r^p}=ar^{q-1}$; but from the first of these,

$$\frac{1}{r^p}=\left(\frac{n}{m}\right)^{\frac{p}{2q}} \therefore \frac{m}{r^p}=m\left(\frac{n}{m}\right)^{\frac{p}{2q}}=ar^{q-1}, \text{ the } q\text{th term.}$$

The next two conditions are

$$P=ar^{p-1},\ Q=ar^{q-1} \therefore \frac{P}{Q}=\frac{ar^{p-1}}{ar^{q-1}}=r^{p-q} \therefore r=\left(\frac{P}{Q}\right)^{\frac{1}{p-q}}$$

$$a=\frac{P}{r^{p-1}}=P\div\left(\frac{P}{Q}\right)^{\frac{p-1}{p-q}} \therefore ar^{n-1}=P\left(\frac{P}{Q}\right)^{\frac{n-1}{p-q}}\div\left(\frac{P}{Q}\right)^{\frac{p-1}{p-q}};$$

$$\text{that is, } ar^{n-1}=P\left(\frac{P}{Q}\right)^{\frac{n-p}{p-q}}=\left(\frac{P^{n-q}}{Q^{n-p}}\right)^{\frac{1}{p-q}}, \text{ the } n\text{th term.}$$

(24.) The conditions are that $\frac{ma+nb}{m+n}=\frac{m+n}{2}=\sqrt{ab}$

$$\left.\begin{aligned}\therefore m+n=2\sqrt{ab} \therefore ma+nb&=2ab\\ \text{and}\quad mb+nb&=2b\sqrt{ab}\end{aligned}\right\}. \quad \text{Subtract}$$

$$\therefore m(a-b)=2b(a-\sqrt{ab})=2b\sqrt{a}(\sqrt{a}-\sqrt{b})$$

$$\therefore m=\frac{2b\sqrt{a}(\sqrt{a}-\sqrt{b})}{a-b}=\frac{2b\sqrt{a}}{\sqrt{a}+\sqrt{b}}.$$

In like manner, by eliminating m instead of n, we should find

$$n=\frac{2a\sqrt{b}}{\sqrt{a}+\sqrt{b}},$$

as is sufficiently obvious without executing any work, for the foregoing equations remain unaltered, though m and n be interchanged, provided we interchange a and b.

(25.) $S=a+ar+ar^2+$ &c., ad. inf. $=\frac{a}{1-r}$,

to n terms $=a\frac{r^n-1}{r-1}$

$s^2=a^2+a^2r^2+a^2r^4+$ &c., ad. inf. $=\frac{a^2}{1-r^2}$

$$S^2-s^2=\frac{a^2(1+r)-a^2(1-r)}{(1-r)^2(1+r)},\quad S^2+s^2=\frac{a^2(1+r)+a^2(1-r)}{(1-r)^2(1+r)}$$

$$\therefore \frac{S^2-s^2}{S^2+s^2}=\frac{a^2r}{a^2}=r, \text{ and } S\{1-r^n\}=a\frac{r^n-1}{r-1}$$

$$\therefore \text{Sum to } n \text{ terms } =S\left\{1-\left(\frac{S^2-s^2}{S^2+s^2}\right)^n\right\}.$$

Harmonical Progression (Page 138).

(1.) The extremes of the harmonic series being 2 and 4, those of the arithmetic series are $\frac{1}{2}$ and $\frac{1}{4}$.

$$l=a+(n-1)d \therefore \frac{1}{4}=\frac{1}{2}+3d \therefore d=-\frac{1}{12}.$$

Therefore the two arithmetic means are $\frac{5}{12}$ and $\frac{1}{3}$: hence the harmonic means are $\frac{12}{5}=2\frac{2}{5}$, and 3.

(2.) The extremes of the harmonic series are 4 and $\frac{12}{7}$; those of the corresponding arithmetic series are $\frac{1}{4}$ and $\frac{7}{12}$

$$l=a+(n-1)d \therefore \frac{7}{12}=\frac{1}{4}+4d \therefore d=\frac{1}{12}.$$

Therefore the three arithmetic means are $\frac{1}{3}$, $\frac{5}{12}$, and $\frac{1}{2}$; hence the harmonic means are 3, $2\frac{2}{5}$, and 2.

(3.) To find the arithmetic mean, we have merely to take half the sum of the extremes; that is, $\frac{1}{2}(3\frac{3}{8}+1\frac{1}{2})=2\frac{7}{16}$. The harmonic mean is the reciprocal of $\frac{1}{2}\left(\frac{8}{27}+\frac{2}{3}\right)$; which is

$\frac{27}{13}=2\frac{1}{13}$; and the geometric mean is $\sqrt{\left(\frac{27}{8}\cdot\frac{3}{2}\right)}=$

$$\sqrt{\frac{81}{16}}=\pm\frac{9}{4}=2\tfrac{1}{4}, \text{ or } -2\tfrac{1}{4}$$

(4.) The extremes of the arithmetic series are 5 and 11. $11=5+3d$ ∴ $d=2$ ∴ the arith. means are 7 and 9 ∴ the harmonic means are $\frac{1}{7}$ and $\frac{1}{9}$.

(5.) The corresponding arithmetic series is $\frac{2}{3}$, $\frac{7}{15}$, $\frac{4}{15}$, where the common difference is $\frac{2}{3}-\frac{7}{15}=\frac{3}{15}$; hence the continuation of the arithmetic series, to the extent of three terms each, is $\frac{19}{15}$, $\frac{16}{15}$, $\frac{13}{15}$, and $\frac{1}{15}$, $-\frac{2}{15}$, $-\frac{5}{15}$.

Consequently, the required terms of the harmonic series are

$$\frac{15}{19},\ \frac{15}{16},\ 1\tfrac{2}{13} \text{ and } 15,\ -7\tfrac{1}{2},\ -3$$

(6.) Let the numbers be x and $x+8$: their harmonic mean is the reciprocal of $\frac{1}{2}\left(\frac{1}{x}+\frac{1}{x+8}\right)=\frac{5}{9}$, by the question,

$$\therefore x+8+x=\frac{10}{9}x(x+8) \ \therefore\ 10x^2+62x=72$$

$$\therefore x^2+\frac{62}{10}x+\left(\frac{31}{10}\right)^2=\frac{72}{10}+\frac{31^2}{100}=\frac{1681}{100}$$

$$\therefore x+\frac{31}{10}=\pm\frac{41}{10}\ \therefore\ x=1, \text{ or } -7\tfrac{1}{5}.$$

Hence the numbers are either 1 and 9, or $-7\frac{1}{5}$ and $\frac{4}{5}$.

(7.) Let the first term be x; then the series is

$$x,\ 2,\ 4\tfrac{4}{35}-x,$$

and the corresponding arithmetic series is

$$\frac{1}{x},\ \frac{1}{2},\ \frac{1}{4\frac{4}{35}-x}\ \therefore\ \frac{1}{x}+\frac{35}{144-35x}=1$$

$$\therefore 144-35x+35x=144x-35x^2$$

$$\therefore 35x^2-144x=-144$$

$$\therefore x^2-\frac{144}{35}x+\left(\frac{72}{35}\right)^2=\left(\frac{72}{35}\right)^2-\frac{144}{35}=\frac{144}{35^2}$$

$$\therefore x-\frac{72}{35}=\pm\frac{12}{35} \therefore x=2\tfrac{2}{5}, \text{ or } 1\tfrac{5}{7}.$$

Hence the series is $2\frac{2}{5}$, 2, $1\frac{5}{7}$, or this reversed.

(8.) Let x and y be the numbers, then the arithmetic mean is $\frac{1}{2}(x+y)$, and consequently the harmonic mean is

$$12\tfrac{5}{7}-\tfrac{1}{2}(x+y)\text{; also,}$$

$$\tfrac{1}{2}(x+y)-12\tfrac{5}{7}+\tfrac{1}{2}(x+y)=x+y-12\tfrac{5}{7}=1\tfrac{2}{7}.\ldots(\text{A}).$$

Now, since an harmonic mean is twice the product divided by the sum,

$$\therefore \frac{2xy}{x+y}=12\tfrac{5}{7}-\tfrac{1}{2}(x+y).\ldots(\text{B}).$$

From (A), $x+y=14 \therefore$ (B) is $\frac{2xy}{14}=12\frac{5}{7}-7=5\frac{5}{7}$

$$\therefore xy=40 \therefore x=10,\ y=4.$$

(9.) The geometric mean is $\sqrt{xy}$, and the harmonic mean is $\frac{2xy}{x+y}$; and by the question, $\frac{2xy}{x+y}=\frac{n}{m}\sqrt{xy}$

$$\therefore \frac{2\sqrt{xy}}{x+y}=\frac{n}{m} \therefore \frac{x^2+2xy+y^2}{4xy}=\frac{m^2}{n^2}\text{; that is, } \tfrac{1}{4}\left(\frac{x}{y}+\frac{y}{x}\right)=\frac{m^2}{n^2}-\tfrac{1}{2}$$

Put $\frac{x}{y}=z \therefore z+\frac{1}{z}=\frac{4m^2-2n^2}{n^2} \therefore z^2-\frac{4m^2-2n^2}{n^2}z=-1$

$$\therefore z^2-\frac{4m^2-2n^2}{n^2}+\left(\frac{2m^2-n^2}{n^2}\right)^2=\frac{4m^4-4m^2n^2}{n^4}$$

$$\therefore z-\frac{2m^2-n^2}{n^2}=\frac{2m}{n^2}\sqrt{(m^2-n^2)}$$

$$\therefore z=\frac{x}{y}=\frac{2m^2-n^2+2m\sqrt{(m^2-n^2)}}{n^2}=\frac{m+\sqrt{(m^2-n^2)}}{m-\sqrt{(m^2-n^2)}}$$

that is, $x:y::m+\sqrt{(m^2-n^2)}:m-\sqrt{(m^2-n^2)}$.

Otherwise thus: $\frac{x+y}{2\sqrt{xy}}=\frac{m}{n}$. Adding and subtracting 1,

$$\frac{(\sqrt{x}+\sqrt{y})^2}{2\sqrt{xy}}=\frac{m+n}{n},\quad \frac{(\sqrt{x}-\sqrt{y})^2}{2\sqrt{xy}}=\frac{m-n}{n}.$$

Dividing, $\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}=\sqrt{\frac{m+n}{m-n}}$ $\therefore$ by the principle at p. 37,

$$\frac{\sqrt{x}}{\sqrt{y}}=\frac{\sqrt{(m+n)}+\sqrt{(m-n)}}{\sqrt{(m+n)}-\sqrt{(m-n)}}$$

$$\therefore \frac{x}{y}=\frac{m+\sqrt{(m^2-n^2)}}{m-\sqrt{(m^2-n^2)}}.$$

(10.) This is what is called a *recurring* series: it is a particular case of the series generated by the development of the fraction

$$\frac{1+x}{1-x-x^2}=1+2x+3x^2+5x^3+8x^4+\text{ \&c.}\ldots.. +\frac{\text{Rem.}}{1-x-x^2}.$$

The sum of n terms of this series, in the case in which $x=1$, may be ascertained as follows:

$$\left.\begin{matrix}1+r+r^2+r^3+\ldots\ldots\ldots\ldots r^n\\ -1-r'-r'^2-r'^3-\ldots\ldots.r'^n\end{matrix}\right\}\div(r-r')=$$

$$1+\{r+r'\}+\{r(r+r')+r'^2\}+\{r(r^2+rr'+r'^2)+r'^3\}+\text{\&c.(A)}$$

In order that this may correspond to the series,

$$1+1+2+3+5+8+\text{ \&c.}\ldots..\text{(B)},$$

we must have $r+r'=1, r+r'^2=2$; that is, $r+(1-r)^2=2$

$$\therefore r^2-r=1 \therefore r=\frac{1+\sqrt{5}}{2} \therefore r'=\frac{1-\sqrt{5}}{2}\ldots.\text{(C)}.$$

It is obvious that the conditions

$$r+r'=1,\ r^2+rr'+r'^2=2, \text{ involve also } rr'=-1,$$

which results from subtracting the second from the square of the first. These conditions render the two series (A) and (B) identical; for the first three terms in (A) are thus made to agree with those of (B), and any following term of (A) is the sum of the two immediately preceding terms: thus, taking the fourth term,

$$(1-r')(r^2+rr'+r'^2)+r'^3$$

$$=r^2+rr'+r'^2-rr'(r+r')=\{r^2+rr'+r'^2\}+\{r+r'\}$$

since $rr'=-1$: so that the fourth term is equal to the sum of the two preceding terms; and so of the others. Now, it is easily seen that the sum of n terms of (B), if increased by 1, will be the $n+2$th term of the series (see NOTE below). Consequently, to find the sum of n terms, we have only to

subtract 1 from the $n+2$th term. The $n+2$th term is seen above to be

$$\frac{r^{n+2}-r'^{n+2}}{r-r'} \therefore \text{(C), the sum of } n \text{ terms is}$$

$$S_n=\frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{n+2}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right\}-1.$$

NOTE.—That this property is general for all series in which any term is the sum of the two terms immediately preceding, may be proved as follows:

Let $a_1+a_2+a_3+a_4+\ldots\ldots a_n+a_{n+1}+a_{n+2}+\ldots\ldots$ be any such series: then the conditions are

$$\begin{aligned}
a_2&=0+a_1\\
a_3&=a_1+a_2\\
a_4&=a_2+a_3\\
&\vdots\\
a_n&=a_{n-2}+a_{n-}\\
a_{n+1}&=a_{n-1}+a_n\\
a_{n+2}&=a_n+a_{n1}.
\end{aligned}$$

Hence, by addition and subtraction, we have

$$a_{n+}=a_1+a_1+a_2+a_3+\ldots\ldots a_n.$$

Consequently, if S_n be the sum of n terms of the proposed series,

$$a_{n+2}=S_n+a_1.$$

If the law do not commence till the third term, then

$$a_{n+2}=S_n+a_2.$$

BINOMIAL SURDS (Page 142).

(1.) Assume $\sqrt{(2+\sqrt{3})}=\sqrt{x}+\sqrt{y}$

$\therefore 2+\sqrt{3}=x+y+2\sqrt{xy} \therefore x+y=2$, and $2\sqrt{xy}=\sqrt{3}$

$\therefore 4xy=3 \therefore (x+y)^2-4xy=4-3=1 \therefore x-y=1$

$\therefore x=\frac{3}{2}, y=\frac{1}{2} \therefore \sqrt{(2+\sqrt{3})}=\sqrt{\frac{1}{2}}+\sqrt{\frac{3}{2}}.$

Again: assume $\sqrt{(8+2\sqrt{7})}=\sqrt{x}+\sqrt{y}$

$\therefore 8+2\sqrt{7}=x+y+2\sqrt{xy} \therefore x+y=8$, and $2\sqrt{xy}=2\sqrt{7}$

$\therefore 4xy=28 \therefore (x+y)^2-4xy=64-28=36 \therefore x-y=6$

$\therefore x=7, y=1 \therefore \sqrt{(8+2\sqrt{7})}=1+\sqrt{7}.$

Lastly: assume $(4-\sqrt{7})=\sqrt{x}-\sqrt{y}$ $\therefore 4-\sqrt{7}=x+y-2\sqrt{xy}$

$$\therefore x+y=4 \text{ and } 2\sqrt{xy}=\sqrt{7} \therefore 4xy=7$$

$$\therefore (x+y)^2-4xy=16-7=9 \therefore x-y=3$$

$$\therefore x=\frac{7}{2},\ y=\frac{1}{2} \therefore \sqrt{(4-\sqrt{7})}=\sqrt{\frac{7}{2}}-\sqrt{\frac{1}{2}}.$$

(2.) 1. Assume $\sqrt{(8-2\sqrt{15})}=\sqrt{x}-\sqrt{y}$

$$\therefore 8-2\sqrt{15}=x+y-2\sqrt{xy} \therefore x+y=8, \text{ and } 2\sqrt{xy}=2\sqrt{15}$$

$$\therefore 4xy=60 \therefore (x+y)^2-4xy=64-60=4 \therefore x-y=2$$

$$\therefore x=5,\ y=3 \therefore \sqrt{(8-2\sqrt{15})}=\sqrt{5}-\sqrt{3}.$$

2. Assume $\sqrt{(1-4\sqrt{-3})}=\sqrt{x}-\sqrt{y}$

$$\therefore 1-4\sqrt{-3}=x+y-2\sqrt{xy}$$

$$\therefore x+y=1, \text{ and } 4\sqrt{-3}=2\sqrt{xy} \therefore 4xy=-48$$

$$\therefore (x+y)^2-4xy=1+48=49 \therefore x-y=7$$

$$\therefore x=4,\ y=-3 \therefore \sqrt{(1-4\sqrt{-3})}=2-\sqrt{-3}.$$

3. Assume $\sqrt{\{1+\sqrt{(1-m^2)}\}}=\sqrt{x}+\sqrt{y}$

$$\therefore 1+\sqrt{(1-m^2)}=x+y+2\sqrt{xy}$$

$$\therefore x+y=1, \text{ and } 2\sqrt{xy}=\sqrt{(1-m^2)} \therefore 4xy=1-m^2$$

$$\therefore (x+y)^2-4xy=1-1+m^2=m^2 \therefore x-y=m$$

$$\therefore x=\frac{1+m}{2},\ y=\frac{1-m}{2}$$

$$\therefore \sqrt{\{1+\sqrt{(1-m^2)}\}}=\sqrt{\frac{1+m}{2}}+\sqrt{\frac{1-m}{2}}.$$

4. Here $3\sqrt{3}+2\sqrt{6}=\sqrt{3}\,.\,(3+2\sqrt{2})$.

Assume $\sqrt{(3+2\sqrt{2})}=\sqrt{x}+\sqrt{y}$ $\therefore 3+2\sqrt{2}=x+y+2\sqrt{xy}$

$$\therefore x+y=3, \text{ and } 2\sqrt{xy}=2\sqrt{2} \therefore 4xy=8$$

$$\therefore (x+y)^2-4xy=9-8=1 \therefore x-y=1 \therefore x=2,\ y=1$$

$$\therefore \sqrt{(3\sqrt{3}+2\sqrt{6})}=(1+\sqrt{2})\sqrt[4]{3}.$$

(3.) 1. Assume $\sqrt{(14+8\sqrt{3})}=\sqrt{x}+\sqrt{y}$

$$\therefore 14+8\sqrt{3}=x+y+2\sqrt{xy}$$

$$\therefore x+y=14, \text{ and } 2\sqrt{xy}=8\sqrt{3} \therefore 4xy=192$$

$$\therefore (x+y)^2-4xy=196-192=4 \therefore x-y=2$$

$$\therefore x=8,\ y=6 \therefore \sqrt{(14+8\sqrt{3})}=\sqrt{8}+\sqrt{6}=(2+\sqrt{3})\sqrt{2}.$$

Assume $\sqrt{(2+\sqrt{3})}=\sqrt{x}+\sqrt{y}$ $\therefore 2+\sqrt{3}=x+y+2\sqrt{xy}$

$$\therefore x+y=2, \text{ and } 2\sqrt{xy}=\sqrt{3} \therefore 4xy=3$$

$$\therefore (x+y)^2-4xy=4-3=1 \therefore x-y=1$$

$$\therefore x=\frac{3}{2},\ y=\frac{1}{2} \therefore \sqrt{(2+\sqrt{3})}=\frac{\sqrt{3}+1}{\sqrt{2}}$$

$$\therefore \sqrt[4]{(14+8\sqrt{3})}=\sqrt{(2+\sqrt{3})}\sqrt[4]{2}=\frac{\sqrt{3}+1}{\sqrt{2}}\sqrt[4]{2}=\frac{\sqrt{3}+1}{\sqrt[4]{2}}.$$

2. $\frac{17}{3}-4\sqrt{2}=\frac{1}{3}(17-12\sqrt{2}).$

Assume $\sqrt{(17-12\sqrt{2})}=\sqrt{x}-\sqrt{y}$

$$\therefore 17-12\sqrt{2}=x+y-2\sqrt{xy} \therefore x+y=17, \text{ and } 2\sqrt{xy}=12\sqrt{2}$$

$$\therefore 4xy=288 \therefore (x+y)^2-4xy=289-288=1 \therefore x-y=1$$

$$\therefore x=9,\ y=8 \therefore \sqrt{(17-12\sqrt{2})}=3-\sqrt{8}.$$

Assume $\sqrt{(3-\sqrt{8})}=\sqrt{x}-\sqrt{y} \therefore 3-\sqrt{8}=x+y-2\sqrt{xy}$

$$\therefore x+y=3, \text{ and } 2\sqrt{xy}=\sqrt{8} \therefore 4xy=8$$

$$\therefore (x+y)^2-4xy=9-8=1 \therefore x-y=1, \therefore x=2,\ y=1$$

$$\therefore \sqrt{(3-\sqrt{8})}=\sqrt{2}-1 \therefore \sqrt{\left(\frac{17}{3}-4\sqrt{2}\right)}=(\sqrt{2}-1)\sqrt[4]{\frac{1}{3}}.$$

INDETERMINATE COEFFICIENTS (Page 145).

(1.) Assume $\frac{x^2}{(x^2-1)(x-2)}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{x-2}=$

$$\frac{A(x-1)(x-2)+B(x+1)(x-2)+C(x^2-1)}{(x^2-1)(x-2)}$$

$$\therefore x^2=(A+B+C)x^2-(3A+B)x+2(A-B)-C$$

$$\therefore A+B+C=1,\ 3A+B=0,\ 2(A+B)-C=0.$$

From the second of these, $B=-3A$ $\therefore$ from the third, $C=8A$, so that the first is

$$A-3A+8A=1 \therefore A=\frac{1}{6} \therefore B=-\frac{1}{2}, C=\frac{4}{3}.$$

Hence the fractions are

$$\frac{1}{6(x-1)}-\frac{1}{2(x-1)}+\frac{4}{3(x-2)}.$$

(3.) Assume $\frac{1+2x}{1-x-x^2}=1+Ax+Bx^2+Cx^3+$ &c,

Multiplying by $1-x+x^2$, we have

$$\left.\begin{array}{r}1+Ax+Bx^2+Cx^3+Dx^4+\&c.\\ -\ x-Ax^2-Bx^3-Cx^4-\&c.\\ -x^2-Ax^3-Bx^4-\&c.\end{array}\right\}=1+2x$$

$$\therefore A-1=2 \therefore A=3;\ B-A-1=0 \therefore B=4;$$
$$C-B-A=0 \therefore C=7$$
$$D-C-B=0 \therefore D=11, \&c,$$

where each coefficient after the second is the sum of the two immediately preceding coefficients;

$$\therefore \frac{1+2x}{1-x-x^2}=1+3x+4x^2+7x^3+11x^4+\&c.$$

(3.) Assume $\sqrt{(1+x)}=1+Ax+Bx^2+Cx^3+Dx^4+\&c.$

$$\text{Squaring } 1+x=\left.\begin{array}{r}1+Ax+Bx^2+Cx^3+Dx^4+\&c.\\ Ax+A^2x^2+ABx^3+ACx^4+\&c.\\ Bx^2+ABx^3+B^2x^4+\&c.\\ Cx^3+ACx^4+\&c.\\ Dx^4+\&c.\end{array}\right\}$$

$$\therefore 2A=1 \therefore A=\frac{1}{2}$$

$$2B+A^2=0 \therefore B=-\frac{A^2}{2}=-\frac{1}{8}$$

$$2C+2AB=0 \therefore C=-AB=\frac{1}{16}$$

$$2D+2AC+B^2=0 \therefore D=-\frac{2AC+B^2}{2}=-\frac{5}{8.16}$$

$$\therefore \sqrt{(1+x)}=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4+\&c.,$$

$$\text{or }=1+\frac{1}{2}x-\frac{1}{2.4}x^2+\frac{3}{2.4.6}x^3-\frac{3.5}{2.4.6.8}x^4+\&c.$$

(4.) For the method of working this example, see the APPENDIX to the Algebra, p. 174.

BINOMIAL THEOREM (Page 151).

(2.) Here, only the first three of the coefficients need be computed; namely,

$$1,\ -5,\ \frac{5.4}{2};\ \text{so that}$$

$$(a-x)^5=a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5.$$

(3.) The first four coefficients are 1, 7, $\frac{7.6}{2}$, $\frac{21.5}{3}$

$\therefore\ (1-x)^7=1-7x+21x^2-35x^3+35x^4-21x^5+7x^6-x.^7$

(4.) The three leading coefficients are 1, 5, 10, as in Ex. 1;

$$\therefore\ (2x+1)^5=(2x)^5+5(2x)^4+10(2x)^3+10(2x)^2+5(2x)+1$$
$$=32x^5+80x^4+80x^3+40x^2+10x+1.$$

(5.) $$(c+x)^{-2}=c^{-2}-2c^{-3}x+\frac{2\,.\,3}{2}c^{-4}x^2-\frac{3\,.\,4}{3}c^{-5}x^3+\ \&c.$$
$$=\frac{1}{c^2}\left(1-\frac{2x}{c}+\frac{3x^2}{c^2}-\frac{4x^3}{c^3}+\ \&c.\right)$$

(6.) $$(1-x^3)^{\frac{1}{3}}=1-\frac{1}{3}x^3+\frac{-\frac{1}{3}\cdot\frac{2}{3}}{2}x^6-\frac{\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{5}{3}}{2\,.\,3}x^9+\ \&c.$$
$$=1-\frac{x^3}{3}-\frac{x^6}{9}-\frac{5x^9}{81}-\ \&c.$$

Variations, Permutations, and Combinations

(Page 158).

(7.) By permutation, Art. XXXII., $p=1\,.\,2\,.\,3\,.\,4\,.\,5=120$.

(8.) As 2 recurs *twice*, 3 *three* times, and 4 *four* times, we have by the general formula, Art. XXXIII.,

$$p=\frac{1\,.\,2\,.\,3\,.\,4\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,10}{1\,.\,2\,.\,1\,.\,2\,.\,3\,.\,1\,.\,2\,.\,3\,.\,4}=\frac{5\,.\,6\,.\,7\,.\,8\,.\,9\,.\,10}{1\,.\,2\,.\,1\,.\,2\,.\,3}=$$
$$5\,.\,7\,.\,4\,.\,9\,.\,10=12600.$$

(9.) Here $V_4=8\,.\,7\,.\,6\,.\,5=1680$, the number of changes.

(10.) The number of variations of $2n+1$ things taken $n-1$ together is

$$(2n+1)(2n)(2n-1)\ldots.\ \{2n+1-(n-2)\},$$

and of $2n-1$ things taken n together,

$$(2n-1)(2n-2)\ldots.\ \{2n-1-(n-1)\}.\quad \text{(Art. XXXII.)}$$

The last factor in the first of these expressions is $n+3$; the last factor in the other is n. Hence, reversing the factors and dividing, we have

$$\frac{(n+3)(n+4)\ldots(2n-1)(2n)(2n+1)}{n(n+1)(n+2)\ldots(2n-1)}=\frac{2n(2n+1)}{n(n+1)(n+2)}.$$

Hence by the question,

$$\frac{2(2n+1)}{(n+1)(n+2)}=\frac{3}{5} \therefore 20n+10=3n^2+9n+6$$

$\therefore 3n^2-11n=4$. Solving this quadratic, we have $n=4$.

(11.) By Art. XXXV., the number of combinations of n things taken r together is

$$\frac{n(n-1)(n-2)\ldots\ldots\{n-(r-1)\}}{1\,.\,2\,.\,3\ldots\ldots r}.$$

In the present example, $n=52$ and $r=13$, and

$$\frac{52\,.\,51\,.\,50\ldots\ldots 40}{1\,.\,2\,.\,3\ldots\ldots 13}=635013559600.$$

(12.) The general expression at Art. XXXV., when $n=50$ and $r=4$, as in the present example, is

$$\frac{50\,.\,49\,.\,48\,.\,47}{1\,.\,2\,.\,3\,.\,4}=50\,.\,49\,.\,47\,.\,2=230300.$$

(13.) The expression for the number of combinations that can be formed out of n things by taking them first singly, then two at a time, then three at a time, and so on, is 2^n-1 (Art. XXXIV.) Hence the number in the present case is

$$2^{16}-1=256^2-1=65535.$$

(14.) The given conditions here are

$$(m+n)(m+n-1)=56$$
$$(m-n)(m-n-1)=12;$$
that is, $$(m+n)^2-(m+n)=56$$
$$(m-n)^2-(m-n)=12.$$

Solving these quadratics, we have

$$m+n=8,\ m-n=4 \therefore m=6,\ n=2.$$

The number of combinations of 2 things out of 6 is $\frac{6\,.\,5}{1\,.\,2}=15.$

(15.) By Art. XXXIV., the permutation of n things taken m together, when repetitions are allowed, is n^m. In the present case, $n=12$, $m=5$; and $12^5=144^2\times12=248832$.

(16.) Let each of the p things be represented by a, each of the q things by b, of the r things by c, &c. Then, as the combinations are to be taken 1, 2, 3, &c., and all together, it

is plain there will be as many of them as there are divisors of $a^p b^q c^r$, &c. The question, therefore, is, to find the number of divisors of this expression, for each quotient will be a combination, except the quotient 1, arising from dividing the expression by itself. The divisors, not involving different letters, may be grouped as follows, the unit divisor being, for the present, repeated:—

$1,\ a,\ a^2,\ a^3, \dots\ a^p$	the number of which is	$p+1$
$1,\ b,\ b^2,\ b^3, \dots\ b^q$	,, ,,	$q+1$
$1,\ c,\ c^2,\ c^3, \dots\ c^r$	,, ,,	$r+1$
&c.		&c.

In addition to these, there will be the divisors furnished by those several terms of the product

$(1+a+a^2+\dots a^p)(1+b+b^2+\dots b^q)(1+c+c^2+\dots c^r)$&c., which involve two or more different letters.

All the terms of this product will, of course, include those grouped above, without any repetition of the unit, so that the total number of these terms will be the total number of combinations, and one more as noticed already. Now, the number of terms in the product of the $p+1$ quantities $1+a+a^2+$ &c. by the $q+1$ quantities $1+b+b^2+$ &c., is evidently $(p+1)(q+1)$; and the number of terms furnished by multiplying these $(p+1)(q+1)$ terms by the $r+1$ quantities $1+c+c^2+$ &c. is $(p+1)(q+1)(r+1)$, and so on. Hence, the total number of divisors is $(p+1)(q+1)(r+1)$ &c., and the number of combinations is therefore

$$(p+1)(q+1)(r+1) \text{ \&c.} -1.$$

(17.) By Art. XXXVI., the number of selections is
$$40 \, . \, 42 \, . \, 45 \, . \, 50 = 3780000.$$

Compound Interest and Annuities (Page 163).

(4.) By Art. XL., $M=PR^n$, where M is the amount, P the principal, and R the amount of £1 for a single period; that is, $R=1+r$, r being the interest of £1 for that period: also, $n=$ the number of periods.

In the present example,

$$P=800,\ r=\cdot05 \ \therefore\ R=1\cdot05 \text{ and } n=9;$$

hence the expression for the amount is $M=800\times1\cdot05^9$, or in logarithms, $\log M=\log 800+9\log 1\cdot05$

$$9 \log 1{\cdot}05 = {\cdot}0211893 \times 9 = {\cdot}1807037$$
$$\log 800 \quad . \quad . \quad . \quad . \quad . \quad 2{\cdot}9030900$$

$$\log 1241{\cdot}063 \quad . \quad . \quad . \quad 3{\cdot}0837937$$

$$\therefore \text{M} = \pounds 1241{\cdot}063$$
$$\text{P} = 800$$

$$441{\cdot}063 \therefore \pounds 441\ 1s.\ 3d. = \text{comp. int.}$$
$$20$$
$$1{\cdot}26$$
$$12$$
$$3{\cdot}12$$

(5.) The logarithmic formula may be written

$$\log \text{M} = \log (\text{R}^n - 1) + \log \text{A} - \log \frac{100r}{100}$$
$$= \log (1{\cdot}03^{18} - 1) + \log 178 - \log 3 + 2,$$

r being $= {\cdot}03$, and $\text{A} = \pounds 178$, because the interest and the annuity are both payable half-yearly

$$\log 1{\cdot}03 = {\cdot}0128372$$
$$18$$
$$1026976$$
$$128372$$

$$\log 1{\cdot}03^{18} = {\cdot}2310696$$
Corresponding number $1{\cdot}702431 = 1{\cdot}03^{18}$
$$-1$$
$${\cdot}702431 = 1{\cdot}03^{18} - 1$$

$$\log (1{\cdot}03^{18} - 1) = \bar{1}{\cdot}8466037$$
$$\log 178 = 2{\cdot}2504200$$
$$2 - \log 3 = 1{\cdot}5228787$$

$$\log \text{M} = 3{\cdot}6199024 \therefore \text{M} = \pounds 4167{\cdot}76$$
$$= 4167\ 15s.\ 2\tfrac{1}{4}d.$$

(6.) Each quarterly payment is £17·5, and the quarterly interest is £1·25 per cent.; so that $\frac{1\cdot25}{100}=r$. The formula for V is

$$\begin{aligned}\log \text{V} &= \log (1-\text{R}^{-n}) + \log \text{A} - \log r\\ &= \log (1-1\cdot0125^{-20}) + \log 17\cdot5 - \log 1\cdot25 + 2\end{aligned}$$

$$\begin{array}{rl}
\log 1\cdot0125^{-1} = & \bar{1}\cdot9946050 \\
 & \quad\quad\quad\quad 2 \\ \hline
 & 19.892100 \\
 & -20 \\ \hline
\log.\ 1\cdot0125^{-20} = & \bar{1}\cdot892100 \\
\text{Corresponding number} & \cdot78001 = 1\cdot0125^{-20} \\
 & 1 \\ \hline
 & \cdot21999 = 1-1\cdot0125^{-20}
\end{array}$$

$$\begin{array}{ll}
\log (1-1\cdot0125^{-20}) & = \bar{1}\cdot3424029 \\
\log 17\cdot5 & = 1\cdot2430380 \\
2-\log 1\cdot25 & = 1\cdot9030900 \\ \hline
\log \text{V} & = 2\cdot4885309 \quad \therefore \text{V} = £307\cdot986
\end{array}$$

$$\begin{array}{r}
£307\cdot986 \\
20 \\ \hline
19\cdot72 \\
12 \\ \hline
8\cdot64 \\
4 \\ \hline
2\cdot56 \\ \hline
\end{array}$$

Hence the present value is £307 19*s.* $8\frac{1}{2}$*d.*

NOTE.—When the interest is said to be p per cent. per annum, payable half-yearly or quarterly, as in the foregoing examples, the meaning intended is, that $\frac{1}{2}p$ or $\frac{1}{4}p$ is to be regarded as the interest for one period. In reality, therefore, the interest per cent. per *annum* is more than p: it is, in fact,

$$100\left\{\left(1+\frac{p}{200}\right)^2-1\right\},\ \text{or}\ 100\left\{\left(1+\frac{p}{400}\right)^4-1\right\},$$

according as the payments are made half-yearly or quarterly.

(7.) The formula for v, in the case of a deferred annuity, is

$$\log v = \log (1 - R^{-n}) + \log A - m \log R - \log r$$

where n is the number of years the annuity is to continue, and m the number of years it is deferred. In the present example,

$$\log v = \log (1 - 1{\cdot}05^{-20}) + \log 1000 - 5 \log 1{\cdot}05 - \log 5 + 2$$
$$= \log (1 - 1{\cdot}05^{-20}) - 5 \log 1{\cdot}05 - \log 5 + 5$$
$$= \log (1 - 1{\cdot}05^{-20}) + 5(1 - \log 1{\cdot}05) - \log 5$$

$\log 1{\cdot}05^{-1} = \bar{1}{\cdot}9788107 \;\therefore\; 1 - \log 1{\cdot}05 = {\cdot}9788107$

$$\begin{array}{rr}
\bar{1}{\cdot}9788107 & {\cdot}9788107 \\
20 & 5 \\
\hline
19{\cdot}576214 & 4{\cdot}8940535 \\
-20 & \\
\hline
\end{array}$$

$$\begin{array}{lr}
\log 1{\cdot}05^{-20} = & \bar{1}{\cdot}576214 \\
\text{Corresponding number} & {\cdot}37689 \\
 & 1 \\
\hline
 & {\cdot}62311 = 1 - 1{\cdot}05^{-20} \\
\hline
\end{array}$$

$$\begin{array}{rl}
\log (1 - 1{\cdot}05^{-20}) = & \bar{1}{\cdot}7945647 \\
5(1 - \log 1{\cdot}05) = & 4{\cdot}8940535 \\
\text{Arith. comp. } \log 5 = & {\cdot}3010300 \\
\hline
\end{array}$$

$$\log v = 3{\cdot}9896482 \;\therefore\; v = £9764{\cdot}46$$

$$\begin{array}{r}
{\cdot}46 \\
20 \\
\hline
9 \cdot 2 \\
12 \\
\hline
2 \cdot 4 \\
\hline
\end{array}$$

Hence the present value is £9764 9*s.* 2½*d.*

NOTE.—It may be well to remind the learner, that in such operations as these, when the results involve thousands of pounds, the pence in those results may differ by a penny or two from those which may be obtained by working without logarithms. It must be remembered, that in the tables the decimals of the logarithms after the seventh place are rejected; so that in multiplying a logarithm by a high number, as also in combining several together, the seventh decimal of the result will in general be affected with error. Moreover, the

numbers whose logarithms appear in the tables, do not extend beyond five places of figures; a sixth figure may be found by proportion, but a seventh, from the cause stated above, is seldom to be depended upon. If by aid of tables carried to a greater number of figures and places of decimals, we had found $v = 9764{\cdot}4642\ldots$, the above result would have been increased by one penny: if the third and fourth decimals had been 63, the result would have been increased by $1\frac{1}{2}d$.

(8.) The formula for the value v of the lease is

$$\log v = \log(1 - R^{-n}) + \log A - \log r$$

$$\therefore \log A = \log v + \log r - \log(1 - R^{-n})$$

$$= \log 100 + \log \frac{5{\cdot}5}{100} - \log(1 - 1{\cdot}055^{-55\frac{1}{4}})$$

$$= \log 5{\cdot}5 - \log(1 - 1{\cdot}055^{-55\frac{1}{4}})$$

$$\begin{array}{r}
\log 1{\cdot}055^{-1} = \bar{1}{\cdot}9767475 \\
55\frac{1}{4} \\
\hline
48837375 \\
48837375 \\
2441869 \\
\hline
53{\cdot}9652994 \\
-55{\cdot}25
\end{array}
\qquad ☞\ -55{\cdot}25 = \overline{56}{\cdot}75$$

$$\log 1{\cdot}055^{-55\frac{1}{4}} = \bar{2}{\cdot}7152994$$

Corresponding number $\cdot 0519158 = 1{\cdot}055^{-55\frac{1}{4}}$

$$\begin{array}{r}
1 \\
\hline
\cdot 9480842 = 1 - 1{\cdot}055^{-55\frac{1}{4}}
\end{array}$$

$$\begin{array}{r}
\log(1 - 1{\cdot}055^{-55\frac{1}{4}}) = \bar{1}{\cdot}9768469 \\
\log 5{\cdot}5 = \phantom{\bar{1}}{\cdot}7403627 \\
\hline
\log A = \phantom{\bar{1}}{\cdot}7635158
\end{array}
\therefore A = £5{\cdot}80117$$

$$= £5\ 16s.$$

(9.) The present value of a perpetual annuity of £A per annum is $v = \frac{A}{r}$: the expression for the perpetuity, in this ex-

ample, is $V = \frac{100}{r}$. The present value of an annuity to continue n years is $V = A\frac{1-R^{-n}}{r}$, which, in the present example, is $V = 100\frac{1-R^{-60}}{r}$. The interest is to be regarded as 5 per cent., so that the value of the perpetuity is £$\frac{100}{\cdot 05}$£$=2000$; that is, 20 years' purchase. For the value of the limited annuity, we proceed as follows:

$$
\begin{array}{rll}
\log 1{\cdot}05^{-1} = & \bar{1}{\cdot}9788107 & \\
 & \phantom{58{\cdot}7286}60 & \\
\hline
 & 58{\cdot}728642 & \\
 & -60 & \\
\hline
\log 1{\cdot}05^{-60} = & \bar{2}{\cdot}728642 & \\
\text{Corresponding number} & {\cdot}0535355 = 1{\cdot}05^{-60} & \\
 & 1 & \\
\hline
1 - 1{\cdot}05^{-60} = & {\cdot}9464645 & \\
 & 2000 = \frac{100}{\cdot 05} & \\
\hline
\text{Value of annuity for 60 years} & 1892{\cdot}9290 & \\
\text{Value of perpetuity} & 2000 & \\
\hline
\text{Difference} & 107{\cdot}071 = £107\ 1s.\ 5d., & \\
\hline
\end{array}
$$

the worth of the freehold above that of the leasehold.

(10.) The formula for V in the case of a deferred or reversionary annuity is

$$\log V = \log (1 - R^{-n}) + \log A - m \log R - \log r$$

$$= \log (1 - 1{\cdot}05^{-14}) + \log 50 - 7 \log 1{\cdot}05 - \log 5 + 2;$$

$$\text{or, since } \log 50 = \log 5 + \log 10 = \log 5 + 1$$

$$\log V = \log (1 - 1{\cdot}05^{-14}) - 7 \log 1{\cdot}05 + 3$$

$$\begin{array}{rlr}
\log 1{\cdot}05^{-1} = & \bar{1}{\cdot}9788107 & \log 1{\cdot}05 = {\cdot}0211893 \\
& 14 & 7 \\
\hline
& 39152428 & {\cdot}1483251 \\
& 9788107 & \\
\hline
& 13{\cdot}7033498 & \\
& -14 & \\
\hline
\log 1{\cdot}05^{-14} = & \bar{1}{\cdot}7033498 & \\
\text{Corresponding number} & {\cdot}505068 & \\
& 1 & \\
\hline
& {\cdot}494932 = 1 - 1{\cdot}05^{-14} & \\
\hline
\log(1 - 1{\cdot}05^{-14}) = & \bar{1}{\cdot}6945456 & \\
-7 \log 1{\cdot}05 = - & {\cdot}1483251 & \\
& 3 & \\
\hline
\log v = & 2{\cdot}5462205 \therefore v = £351{\cdot}739 & \\
& 20 & \\
& 14{\cdot}78 & \\
& 12 & \\
& 9{\cdot}36 &
\end{array}$$

Hence the purchase money is £351 14*s.* 9*d.*

(11.) The formula for the amount, Art. XL., is

$$M = PR^n \therefore 600 = 518{\cdot}3R^3$$

$$\therefore R^3 = \frac{600}{518{\cdot}3} = 1{\cdot}157625 = 1{\cdot}05^3$$

$$\therefore R = 1{\cdot}05 \therefore \text{the rate is 5 per cent.}$$

(12.) The formula for the amount, as in the preceding example, is $M = PR^n$. In the present case, M is to be equal to

$$2P \therefore 2P = PR^n \therefore R^n = 2 \therefore n = \frac{\log 2}{\log R},$$

which is a general expression for the number of years in which any sum will double itself at a given rate of compound interest. As, in the example, the given rate is 4 per cent., we have

$$n=\frac{\log 2}{\log 1\cdot 04}=\frac{\cdot 30103}{\cdot 01703334}=17\cdot 67299 \text{ years.}$$

(13.) The amount of any sum P in n years is PR^n, and the amount of a yearly sum or annuity, A, accumulating for n years is $A\frac{R^n-1}{R-1}$. In the present example, the original sum P is £20; the yearly investment or annuity, A, is also £20. Hence, the whole amount, if A were paid in the nth year as well as every preceding year, would be

$$PR^n+P\frac{R^n-1}{R-1}.$$

But as the sum P would not be paid in the nth year, merely to be returned without interest, it is clear that after the first investment there are only $n-1$ annual payments: hence the annual amount is

$$PR^n+P\frac{R^n-1}{R-1}-P=PR\frac{R^n-1}{R-1}=20(1\cdot 05)\frac{1\cdot 05^{40}-1}{\cdot 05};$$

that is, Amount $=400(1\cdot 05)(1\cdot 05^{40}-1)$

$$\begin{array}{lr}
\log 1\cdot 05= & \cdot 0211893 \\
 & 40 \\
\hline
 & \cdot 8475772 \\
\text{Corresponding number} & 7\cdot 03999 \\
 & 1 \\
\hline
 & 6\cdot 03999=1\cdot 05^{40}-1 \\
 & 400 \\
\hline
 & 2415\cdot 996 \\
 & 1\cdot 05 \\
\hline
 & 12079980 \\
 & 2415996 \\
\hline
 & £\,2536\cdot 79580 \;\therefore\; \text{Amt.}=£2536\ 16s. \\
 & 20 \\
\hline
 & 15\cdot 916 \\
\hline
\end{array}$$

(14.) The formula for the present value is

$$\log v = \log (1-R^{-n}) + \log A - \log r$$

$$= \log (1-1{\cdot}05^{-5}) + \log 40 - \log \frac{5}{100} \quad ☞ \; 40 \div \frac{5}{100} = 800$$

$$= \log (1-1{\cdot}05^{-5}) + \log 800$$

$$
\begin{array}{rl}
\log 1{\cdot}05^{-1} = & \bar{1}{\cdot}9788107 \\
& \phantom{\bar{1}{\cdot}978810}5 \\
\hline
& 4{\cdot}8940535 \\
& -5 \\
\hline
\log 1{\cdot}05^{-5} = & \bar{1}{\cdot}8940535 \\
\text{Corresponding number} & {\cdot}783526 \; = 1{\cdot}05^{-5} \\
& 1 \\
\hline
& {\cdot}216474 \; = 1 - 1{\cdot}05^{-5} \\
\log 1 - 1{\cdot}05^{-5} = & \bar{1}{\cdot}3354057 \\
\log 800 = & 2{\cdot}9030900 \\
\hline
\log v = & 2{\cdot}2384957 \; \therefore \; v = £173{\cdot}18
\end{array}
$$

$$
\begin{array}{r}
{\cdot}18 \\
20 \\
\hline
3{\cdot}60 \\
12 \\
\hline
7{\cdot}2 \\
\hline
\end{array}
$$

Hence the present value is £173 3*s*. 7*d*.

(15.) The formula for the present value of a perpetuity is

$$v = \frac{A}{r} \; \therefore \; v = \frac{A}{{\cdot}045} = \frac{20A}{{\cdot}9}, \text{ where } A = £79 \; 4s.$$

Hence, we have only to multiply the pounds by 20, to add in the shillings as if they were pounds, to annex a 0 to the result, and to divide by 9; thus:

£79 4*s*. × 20 = 1584; and £15840 ÷ 9 = £1760.

NOTE.—The computer will, in general, find his work shortened by putting his algebraic formulæ respecting annuities in a shape a little different from that in which they are exhibited in the ALGEBRA; thus:

For the amount, $M=\frac{A}{r}(R^n-1)$.......... (I)

Present value, $V=\frac{A}{r}(1-R^{-n})$ .... (II)

Perpetuity, $V=\frac{A}{r}$ .... (III)

Deferred annuity, $V=\frac{A}{r}\{R^{-m}(1-R^{-n})\}$ .. (IV)

As shown in the foregoing solutions, the value of $\frac{A}{r}$ may be readily found without logarithms, and thus two references to the tables saved.

(16.) Let $P_1, P_2, P_3, \ldots P_n$, be the amounts at the end of the first, second, third, &c., years respectively: then, r being the interest of £1 for a year, we shall have, by the question,

$$P_1 = P + rP + \frac{rP}{m}$$

$$= P\left(1 + r + \frac{r}{m}\right)$$

$$P_2 = P_1\left(1 + r + \frac{r}{m}\right)$$

$$P_3 = P_2\left(1 + r + \frac{r}{m}\right)$$

$$\vdots$$

$$P_n = P_{n-1}\left(1 + r + \frac{r}{m}\right).$$

Multiplying these together,

$$P_1P_2P_3 \ldots P_n = PP_1P_2 \ldots P_{n-}\left(1 + r + \frac{r}{m}\right)^n$$

$$\therefore\ P_n = P\left(1 + r + \frac{r}{m}\right)^n.$$

(17.) By the formula (IV.) above, we have

$$V = \frac{20}{\cdot 035}\{1\cdot 035^{-10}(1 - 1\cdot 035^{-11})\}$$

$$\log 1{\cdot}035^{-1} = \bar{1}{\cdot}9850597 \qquad \log 1{\cdot}035 = {\cdot}0149403$$

$$\begin{array}{rr}
\bar{1}{\cdot}9850597 & {\cdot}0149403 \\
11 & 10 \\
\hline
10{\cdot}8356567 & {\cdot}1494030 \\
-11 & \\
\hline
\end{array}$$

$$\log 1{\cdot}035^{-11} = \bar{1}{\cdot}8356567$$

Corresponding number $\quad {\cdot}684947$

$$\begin{array}{r}
{\cdot}684947 \\
1 \\
\hline
.315053 = 1 - 1{\cdot}035^{-11} \\
\hline
\end{array}$$

$$\log (1 - 1{\cdot}035^{-11}) = \bar{1}{\cdot}4983836$$
$$-10 \log 1{\cdot}035^{-10} = -{\cdot}1494030$$
$$\overline{\bar{1}{\cdot}3489806}$$

Corresponding number $\quad {\cdot}223347 \qquad$ ☞ $\dfrac{20}{{\cdot}035} = \dfrac{4000}{7}$

$$\begin{array}{r}
{\cdot}223347 \\
4000 \\
\hline
7)893{\cdot}388 \\
\hline
£\ 127{\cdot}627 \\
20 \\
\hline
12{\cdot}54 \\
12 \\
\hline
6{\cdot}48 \\
\hline
\end{array}$$

$\therefore$ v $= £127\ 12s.\ 6\frac{1}{2}d.$

(18.) The present value of a freehold estate, producing A per annum, is $\text{v} = \frac{\text{A}}{r}$; and the present value of an annuity a for n years is $\frac{a}{r}(1 - \text{R}^{-n})$. If these values are to be equal, we must have

$$\frac{a}{r}(1 - \text{R}^{-n}) = \frac{\text{A}}{r} \quad \therefore\ a = \frac{\text{A}}{1 - \text{R}^{-n}}.$$

In the present case, $\text{A} = 500$

$$\therefore\ 2a = \frac{1000}{1 - 1{\cdot}045^{-20}} \quad \therefore\ \log 2a = 3 - \log (1 - 1{\cdot}045^{-20})$$

$$\begin{array}{rl}
\log 1.045^{-1} = & \bar{1}\cdot9808837 \\
 & \quad 20 \\
\hline
 & 19\cdot617674 \\
 & -20 \\
\hline
\log 1\cdot045^{-20} = & \bar{1}\cdot617674 \\
\text{Corresponding number} & \cdot414643 \\
 & 1 \\
\hline
 & \cdot585357 = 1 - 1\cdot045^{-20} \\
\hline
\log (1 - 1\cdot045^{-20}) = & \bar{1}\cdot7674209 \\
 & 3 \\
\hline
\log 2a = & 3\cdot2325791 \;\therefore\; 2a = £1708\cdot36 \\
 & \qquad\qquad \therefore\; a = £\;854\cdot18
\end{array}$$

$$\begin{array}{r}
20 \\
\hline
3\cdot6 \\
12 \\
\hline
7\cdot2 \\
\hline
\end{array}$$

Hence the annuity is £854 3*s.* 7*d.*

End of the Examples in the Algebra.

APPENDIX.

Miscellaneous Examples (Page 181).

(1.) Let one of the numbers be x; then by the question,

$$8 : 5 :: x : \frac{5x}{8}, \text{ the other number;}$$

$$\text{also, } \frac{5x^2}{8} = 360 \;\therefore\; \frac{x^2}{8} = 72$$

$$\therefore x^2 = 9 \times 8 \times 8 \;\therefore\; x = 3 \times 8 = 24.$$

Hence the numbers are 24 and $\frac{5 \times 24}{8} = 15$.

(2.) It is plain that $\frac{3}{4}$ of the work remains to be done when A is called off. As B alone finishes this in 36 days, he does $\frac{1}{4}$ in 12 days, and, consequently, the whole in 48 days: he thus does $\frac{1}{48}$ in a day. But when both A and B work together, they do $\frac{1}{16}$ in a day: hence, A alone does $\frac{1}{16}-\frac{1}{48}=\frac{1}{24}$ in a day, and therefore he can do the whole in 24 days.

Or, since B alone is 48 days doing what, with the help of A, is done in one-third of the time, it is plain that A works twice as fast as B, and can, therefore, do the work in 24 days, or half the time.

(3.) The coefficient of the second term of a quadratic equation is the sum of its two roots with their signs changed, and the third, or absolute term of the quadratic, is the product of the two roots (see p. 60): hence the required quadratic is $x^2-8x+15=0$.

$$\text{(4.) Since } 2x^2+8x+c=0 \therefore x^2+4x=-\frac{c}{2}$$

$$\therefore x^2+4x+4=4-\frac{c}{2}=0, \text{ when the roots are equal}$$

$$\therefore 4=\frac{c}{2} \therefore c=8, \text{ and } (x+2)^2=0 \therefore x=-2.$$

$$\text{(5.) Put } x^2-8x+22=m \therefore x^2-8x+16=m-6$$

$$\therefore x-4=\sqrt{(m-6)} \therefore x=4\pm\sqrt{(m-6)}.$$

Hence, in order that x may be real, m must not be less than 6.

$$\text{(6.) } (a+b)(b+c)(a+c)=2abc+a^2(b+c)+b^2(a+c)+c^2(a+b)$$
$$=2abc+(a^2+b^2)c+(a^2+c^2)b+(b^2+c^2)a.$$

Now (Algebra, p. 101),

$$a^2+b^2>2ab,\ a^2+c^2>2ac,\ b^2+c^2>2bc$$

$$\therefore (a+b)(b+c)(a+c)>2abc+2abc+2abc+2abc, \text{ or } >8abc.$$

(7.) 1 . 2 . 3 . 4 . 5 . 6 . 7 . 8 . 9 . 10 . 11 . 12=479001600.

(8.) By permutation, Art. XXXIII.,

$$P=\frac{1\cdot 2\cdot 3\ldots\ldots\ldots\ldots n}{1\cdot 2\ldots p\cdot 1\cdot 2\ldots q\cdot 1\cdot 2\ldots r}$$

which, because there are 11 letters, of which 4 are i's, 4 are s's, and 2 are p's, becomes

$$P=\frac{1.2.3.\ldots\ldots..11}{1.2.3.4.1.2.3.4.1.2}=\frac{5.6.7.\ldots..11}{1.2.3.4.1.2}=$$
$$5.7.9.10.11=34650.$$

(9.) See p. 75 of this Key.

(10.) Suppose A began at x hours before 12 o'clock,
then B began at $x-\frac{1}{2}$ hours ,, ,,

It appears by the question, that in these hours half the work was done; so that the work done by B in the 6 hours after noon must have been equal to the work done by A in the x hours before noon, and the work done by A in the $8\frac{3}{4}$ hours after noon, must have been equal to the work done by B in the in the $x-\frac{1}{2}$ hours before noon. Consequently, if A's hourly work be represented by a, and B's by b, we have the two equations

$$ax=6b \text{ and } 8\tfrac{3}{4}a=(x-\tfrac{1}{2})b.$$

Dividing the second by the first,

$$\frac{8\frac{3}{4}}{x}=\frac{x-\frac{1}{2}}{6} \therefore x^2-\frac{x}{2}=\frac{105}{2}$$

$$\therefore x^2-\frac{x}{2}+\frac{1}{16}=\frac{105}{2}+\frac{1}{16}=\frac{841}{16}$$

$$\therefore x-\frac{1}{4}=\frac{29}{4} \therefore x=\frac{30}{4}=7\tfrac{1}{2}.$$

Hence A began at half-past four in the morning.

(11.) Suppose that after the 30 days, the voyage ought to have lasted x days, and that each man, according to this estimate of the time, had been provided with a gallon of water a day: then there were $175x$ gallons for the x days. The number of gallons saved by the deaths was

$$3(1+2+3+\ldots.[x+21])=\frac{3(x+21)}{2}(x+22),$$

since the voyage was protracted 21 days beyond the estimated time: hence, had these gallons been actually added to the stock, there would have been sufficient for the whole time, even if all the men had lived; so that, dividing the whole number of gallons thus increased by the product of the number

of men and number of days, the result must be each man's daily allowance; that is, 1 gallon

$$\therefore \frac{175x+\frac{3x+21}{2}(x+22)}{175(x+21)}=1$$

$$\therefore 175x+\frac{3(x+21)}{2}(x+22)=175x+3675$$

$$\therefore 3(x+21)^2+3(x+21)=7350$$

$$\therefore (x+21)^2+(x+21)=2450$$

$$\therefore (x+21)^2+(x+21)+\tfrac{1}{4}=\frac{9801}{4}$$

$$\therefore x+21+\tfrac{1}{2}=\frac{99}{2} \therefore x=28.$$

Consequently, adding to this the 30 days and the 21 days, the sum 79 days is the whole time of passage.

(12.) $(x+y)^7-(x^7+y^7)=7xy(x^5+y^5)+21x^2y^2(x^3+y^3)+35x^3y^3(x+y).$

Dividing this by $7xy(x+y)$, the quotient is

$$x^4-x^3y+x^2y^2-xy^3+y^4+3xy(x^2-xy+y^2)+5x^2y^2=$$
$$x^4+2x^3y+3x^2y^2+2xy^3+y^4=$$
$$(x^2+xy+y^2)^2.$$

Hence the proposed expression is divisible by this square.

(13.) By Art. XXXVII.,

$${}^{100}C_{96}={}^{100}C_4=\frac{100\,.\,99\,.\,98\,.\,97}{1\,.\,2\,.\,3\,.\,4}=3921225.$$

(14.) Let s be the sum, and s the sum of the squares; then

$$\text{s}=\frac{a(r^n-1)}{r-1},\ s=\frac{a^2(r^{2n}-1)}{r^2-1}$$

$$\therefore \frac{s}{\text{s}}=\frac{a(r^{2n}-1)}{(r+1)(r^n-1)}=\frac{a(r^{2n}+1)}{r+1}=$$
$$a-ar+ar^2-ar^3+\ldots\ldots ar^{n-1},$$

whenever n is *odd*.

(15.) Let the arithmetic progression be

$$a,\ a+d,\ a+2d,\ a+3d,\ a+4d,\ a+5d,\ \&c.;$$

then by the given condition,

$$(a+d)^2=a^2+3ad\ \therefore\ d^2=ad\ldots\ (A).$$

Also the product of the fourth and ninth terms is

$$(a+3d)(a+8d)=a^2+11ad+24d^2;$$

$$\text{that is, (A), } =a^2+10ad+25d^2=(a+5d)^2,$$

and $a+5d$ is the sixth term of the progression.

(16.) The formula for the nth term of a geometric progression being $l=ar^{n-1}$, we have

$$9=ar^3,\ \text{and}\ 15=ar^6.\quad \text{Dividing,}$$

$$\frac{5}{3}=r^3\ \therefore\ r=\sqrt[3]{\frac{5}{3}};\ \text{also, } a=9\div\frac{5}{3}=\frac{27}{5}.$$

Hence the progression is that given in the answer.

(17.) Here $\frac{1}{32}$ is the seventh term of a geometric progression, of which 2 is the first term; therefore, from the formula

$$l=ar^{n-1},\ \text{we have } \frac{1}{32}=2r^6\ \therefore\ r^6=\frac{1}{64}=\frac{1}{2^6}\ \therefore\ r=\pm\frac{1}{2}.$$

Hence the five means are $\pm 1,\ \frac{1}{2},\ \pm\frac{1}{4},\ \frac{1}{8},\ \pm\frac{1}{16}.$

(18.) The formula for the sum is

$$s=\frac{n}{2}\{2a+(n-1)d\}$$

$$\therefore\ 24=\frac{n}{2}\{18-(n-1)2\}\ \therefore\ 48=18n-2n^2+2n$$

$$\therefore\ n^2-10n=-24\ \therefore\ n^2-10n+25=1$$

$$\therefore\ n-5=\pm 1\ \therefore\ n=4 \text{ or } 6.$$

(19.) As many changes can be rung by five bells out of six as by the whole peal; namely,

1 . 2 . 3 . 4 . 5 . 6=720 (see Algebra, p. 153).

(20.) By combinations,

$$^{10}C_6={}^{10}C_4=\frac{10\,.\,9\,.\,8\,.\,7}{1\,.\,2\,.\,3\,.\,4}=10\,.\,3\,.\,7=210.$$

(21.) From the first equation, $b^2x^2+b^2y^2=b^2r^2$. From the second, $b^2y^2=r^4-2ar^2x+a^2x^2$: hence, by substitution,

$$b^2x^2+r^4-2ar^2x+a^2x^2=b^2r^2$$

$$\therefore\ (a^2+b^2)x^2-2ar^2x=r^2(b^2-r^2)$$

$$\therefore\ x^2-\frac{2ar^2}{a^2+b^2}x+\left(\frac{ar^2}{a^2+b^2}\right)^2=\frac{b^2r^2(a^2+b^2-r^2)}{(a^2+b^2)^2}$$

$$\therefore\ x=\frac{ar^2}{a^2+b^2}\pm\frac{br}{a^2+b^2}\sqrt{(a^2+b^2-r^2)}.$$

Changing a into b, and x into y, we have similarly

$$y=\frac{br^2}{a^2+b^2}\mp\frac{ar}{a^2+b^2}\sqrt{(a^2+b^2-r^2)}.$$

The sign of the second term must be opposite to that in the expression for x, in order to justify the second equation.

(22.) Dividing by $\frac{x}{3}$, put $x^2+\frac{3}{2}x+\frac{1}{2}=(x+\text{A})(x+\text{B})$;

that is, $x^2+\frac{3}{2}x+\frac{1}{2}=x^2+(\text{A}+\text{B})x+\text{AB}$.

Equating coefficients, $\text{A}+\text{B}=\frac{3}{2}$, $\text{AB}=\frac{1}{2}$ $\therefore\ 4\text{AB}=2$

$\therefore\ \sqrt{\{(\text{A}+\text{B})^2-4\text{AB}\}}=\text{A}-\text{B}=\pm\frac{1}{2}$ $\therefore\ \text{A}=1$, or $\frac{1}{2}$; $\text{B}=\frac{1}{2}$, or 1

$$\therefore\ x^2+\frac{3}{2}x+\frac{1}{2}=\frac{(x+1)(2x+1)}{2}$$

$$\therefore\ \frac{x^3}{3}+\frac{x^2}{2}+\frac{x}{6}=\frac{x(x+1)(2x+1)}{6}.$$

(23.) Let x be the number of gallons at a shillings, then $d-x$ must be the number at b shillings. The price of the former is ax shillings, and that of the latter $b(d-x)$ shillings: hence, because the worth of the compound is to be cd shillings, we have the equation

$$ax+b(d-x)=cd,\ \text{or}\ ax+bd-bx=cd$$

$\therefore\ (a-b)x=(c-b)d$; $\therefore\ x=\frac{c-b}{a-b}d$, gal. of first,

$\therefore\ d-x=\frac{a-c}{a-b}d$, gal. of second.

(24.) At the first drawing, $\frac{b}{a}$ of the whole cask of wine is taken away, so that $1-\frac{b}{a}$ is what remains: of this quantity, $\frac{b}{a}$ is taken away at the second drawing, so that $1-\frac{b}{a}$ of it is left; that is, $\left(1-\frac{b}{a}\right)^2$. In the third drawing, $\frac{b}{a}$ of this is taken away, so that $1-\frac{b}{a}$ of it is left; that is, the quantity left is $\left(1-\frac{b}{a}\right)^3$: hence, after n drawings, the part of the whole cask of wine left is $\left(1-\frac{b}{a}\right)^n$; that is, it is $\left(\frac{a-b}{a}\right)^n$: in other words, there is this part of the original a gallons of wine left in the cask: but $\left(\frac{a-b}{a}\right)^n$ of a is $\frac{(a-b)^n}{a^{n-1}}$, which, therefore, expresses the number of gallons remaining in the vessel.

(25.) Put $\frac{(a+x)(b+x)}{x}=y \therefore x^2+(a+b)x+ab=yx$

$$\therefore x^2+(a+b-y)x=-ab.$$

Solving this quadratic, we have

$$x=\frac{y-(a+b)}{2}\pm\frac{\sqrt{\{[y-(a+b)]^2-4ab\}}}{2}.$$

Equating the expression under the radical to zero (Appendix, p. 169), the value of x becomes $x=\frac{y-(a+b)}{2}$, and we have the condition

$$[y-(a+b)]^2=4ab \therefore y-(a+b)=2\sqrt{ab};$$

so that the value of x is $x=\sqrt{ab}$, the value sought.

(26.) The general formula for the number of shot in a square pile having n shot in the bottom row is

$$\frac{n(n+1)(2n+1)}{6}=\frac{30 \,.\, 31 \,.\, 61}{6}=5 \,.\, 31 \,.\, 61=9455.$$

(27.) The general formula for the number of shot in a triangular pile having n shot in the bottom row is

$$\frac{n(n+1)(n+2)}{6}=\frac{30\,.\,31\,.\,32}{6}=5\,.\,31\,.\,32=4960.$$

(28.) For the proposed value of x, the fraction takes the form $\frac{0}{0}$: hence, numerator and denominator must be divisible by $x-2$ (Appendix, p. 167). Applying this divisor, the fraction is reduced to $\frac{2x^2-x-6}{x^2+2x-8}$. Putting 2 for x, this also takes the form $\frac{0}{0}$; the terms must, therefore, be also divisible by $x-2$: dividing, therefore, again, the fraction is further reduced to $\frac{2x+3}{x+4}$, which, for $x=2$, becomes $\frac{7}{6}$. This, therefore, is the value of the proposed fraction, when $x=2$.

(29.) By the Binomial Theorem,

$$(a^n-x^n)^{\frac{1}{n}}=a-\frac{1}{n}a^{1-n}x^n+\ldots\ldots,$$

where the powers of x go on increasing;

$$\therefore \frac{x^n}{a-\sqrt[n]{(a^n-x^n)}}=\frac{x^n}{\frac{1}{n}a^{1-n}x^n+\ldots\ldots}=\frac{n}{a^{1-n}+\ldots\ldots}=$$

$$na^{n-1},\text{ when } x=0.$$

(30.) Proceeding as explained in Article I. of the Appendix, the transformation is effected as follows:—

3) 1810

3) 603, remainder $=1$

3) 201 .... $=0$

3) 67 .... $=0$

3) 22 .... $=1$

3) 7 .... $=1$

3) 2 .... $=1$

0 .... $=2$.

Hence, in the ternary scale of notation, the number 1810 is 2111001.

(31.) The polygon has n vertices, and the vertices of every triangle are three of these: hence, there are as many triangles as there are combinations of three things out of n: hence, Algebra, Art. XXXV., the number of triangles is

$$\frac{n(n-1)(n-2)}{1.2.3}=\frac{n(n-1)(n-2)}{6}.$$

(32.) It is obvious that from each vertex $n-3$ lines can be drawn to the other vertices, and since there are n vertices, $n(n-3)$ may be drawn altogether: but every set of $n-3$ is repeated by lines from the $n-3$ vertices in which they terminate: hence the number of diagonals is $\frac{n(n-3)}{2}$.

(33.) If m be the number of arms, and n the number of different positions, then (Algebra, p. 158) the total number of signals is

$$(1+^n)^m-1 \therefore (1+5)^4-1=36^2-1=1295$$

is the number of signals.

(34.) Suppose there were x gallons: then, by the question, the profit on each gallon of the first portion sold was $\frac{1}{4}$ the cost, and the profit on the second portion $\frac{7}{4}$ the cost. Also, the whole profit was $\frac{3}{5}$ the cost;

$$\therefore \frac{1}{4}\left(\frac{3}{4}x+2\right)+\frac{7}{4}\left(\frac{x}{4}-2\right)=\frac{3}{5}x;$$

$$\text{that is, } \frac{5}{8}x-\frac{6}{2}=\frac{3}{5}x \therefore 25x-120=24x \therefore x=120,$$

the number of gallons. It was unnecessary to state the cost price.

(35.) Multiplying by $(x+a)(x+b)$, we have

$$cx+d=\text{A}(x+b)+\text{B}(x+a)$$
$$=(\text{A}+\text{B})x+\text{A}b+\text{B}a$$

$$\therefore \text{A}+\text{B}=c,\ \text{A}b+\text{B}a=d$$
$$\therefore \text{A}b+\text{B}b=bc$$

$$\therefore \text{B}(a-b)=d-bc \quad \therefore \text{B}=\frac{d-bc}{a-b}$$

Interchanging A and B, and a and b, $\text{A}=\dfrac{d-ac}{b-a}$

(36.) Extract the square root of the first member:

$$\begin{array}{ll} & 4x^4-12x^3-35x^2+66x-8303(2x^2-3x-11 \\ & 4x^4 \\ \hline 4x^2-3x) & -12x^3-35x^2 \\ & -12x^3+9x^2 \\ \hline 4x^2-6x-11) & -44x^2+66x-8303 \\ & -44x^2+66x+121 \\ \hline & -8424 \end{array}$$

Hence the expression, if increased by 8424, would be a complete square; namely, $(2x^2-3x-11)^2$; the equation is, therefore, the same as

$$(2x^2-3x-11)^2-8424=0,$$

$$\text{or } \{(2x^2-3x-11)-\sqrt{8424}\}\{(2x^2-3x-11)+\sqrt{8424}\}=0$$

$$\therefore 2x^2-3x-11=\pm\sqrt{8424}=\pm 18\sqrt{26}$$

$$\therefore x^2-\frac{3}{2}x=\frac{11\pm 18\sqrt{26}}{2}$$

$$\therefore x^2-\frac{3}{2}x+\frac{9}{16}=\frac{97\pm 144\sqrt{26}}{16}$$

$$\therefore x=\frac{3\pm\sqrt{(97\pm 144\sqrt{26})}}{4}.$$

(37). This is an immediate inference from Colson's Theorem at p. 152 of the Algebra, because $(a+b)^n$ is the same as $(b+a)^n$.

(38.) By the binomial theorem, the $n+1$th term of the expansion of $(1+x)^{2n}$ is

$$\frac{2n(2n-1)(2n-2)\ldots\ldots(n+1)}{1.2.3\ldots\ldots n}x^n\ldots\ldots(\text{A})$$

$$=\frac{1.2.3....n(n+1)....2n(2n-1)}{(1.2.3....n)^2}x^n$$

$$=\frac{1.3.5....(2n-1)2.4.6....2n}{(1.2.3....n)^2}x^n$$

$$=\frac{1.3.5....(2n-1)1.2.3....n}{(1.2.3....n)^2}(2x)^n$$

$$=\frac{1.3.5....(2n-1)}{1.2.3....n}(2x)^n.... \text{ (B)}.$$

Consequently, putting $x=1$, in (A) and (B), we have

$$(n+1)(n+2)....(n+n)=2^n\times 1.3.5....(2n-1).$$

Again: if $x=1$, in the expansion of $(1+x)^n$, we have

$$(1+1)^n=1+n+\frac{n(n-1)}{1.2}+\frac{n(n-1)(n-2)}{1.2.3}+\text{ \&c.}=2^n$$

$$=\frac{(n+1)(n+2)....(n+n)}{1.3.5....(2n-1)}.... \text{ (A)}$$

(A) when n is odd, $=\dfrac{(n+1)(n+2)....(n+n)}{1.3.5....n(n+2)(n+4)....(2n-1)}$

$$=\frac{(n+1)(n+3)....(n+n)}{1.3.5....n}$$

(A) when n is even,

$$=\frac{(n+1)(n+2)....(n+n)}{1.3.5.....(n-1)(n+1)(n+3)....(2n-1)}$$

$$=\frac{(n+2)(n+4)....(n+n)}{1.3.5....(n-1)}.$$

(39.) Let the quotient arising from dividing any number N by 9 be Q, and let the remainder be R: then $N=9Q+R$. In like manner, for any other number n, we have $n=9q+r$. The product of the two numbers is $Nn=81Qq+9Rq+9Qr+Rr$, every term of which is necessarily divisible by 9, except Rr, the product of the remainders. Consequently, if two numerical factors be each divided by 9, and the remainders R, r, noted, and then the product of these remainders be divided by 9, the remainder arising from this last division must be the same as that arising from the division of the product of the two factors by 9.

What is here shown in reference to the divisor 9 equally

holds for any other divisor; but 9 is the divisor chosen to test the accuracy of a multiplication operation in arithmetic, because of the property that it will supply the same remainder whether the number itself or only the sum of its digits be divided. Thus, if the digits of a number—writing them from right to left—be a, b, c, d, &c., then the value of the number is

$$a+10b+100c+1000d+\text{ \&c.};$$

$$\text{that is, } a+(9+1)b+(99+1)c+(999+1)d+\text{ \&c.},$$

and the sum of the digits is $a+b+c+d+$ &c.

And it is plain, that whichever of these we divide by 9, the remainder must be the same, and so of 3 and 11.

(40.) Applying the method of Indeterminate Coefficients, assume

$$x^n+\frac{1}{x^n}=\left(x+\frac{1}{x}\right)^n+\mathrm{A}_1\left(x+\frac{1}{x}\right)^{n-1}+\mathrm{A}_2\left(x+\frac{1}{x}\right)^{n-2}+\ldots\ \mathrm{A}_n$$

Then, developing the several terms by the Binomial Theorem, we have

$$x^n+\frac{1}{x^n}=$$

$$\begin{array}{r|r|r|r|r|l}
x^n+0 & x^{n-1}+n & x^{n-2}+0 & x^{n-3}+\frac{n(n-1)}{1.2} & x^{n-4}+\quad 0 & x^{n-5}+\text{ \&c.} \\
+\mathrm{A}_1 & +0 & +\mathrm{A}_1(n-1) & +0 & +\mathrm{A}_1\frac{(n-1)(n-2)}{1.2} & \\
 & +\mathrm{A}_2 & +0 & +\mathrm{A}_2(n-2) & +0 & \\
 & & +\mathrm{A}_3 & +0 & +\mathrm{A}_3(n-3) & \\
 & & & +\mathrm{A}_4 & +0 & \\
 & & & & +\mathrm{A}_5 &
\end{array}$$

Equating coefficients of like quantities, there results

$$\mathrm{A}_1=0, \qquad \mathrm{A}_2=-n$$

$$\mathrm{A}_3=0, \qquad \mathrm{A}_4=\frac{n(n-3)}{1.2}$$

$$\mathrm{A}_5=0 \qquad \mathrm{A}_6=-\frac{n(n-4)(n-5)}{1.2.3},\text{ \&c., \&c.}$$

$$\therefore\ x^n+\frac{1}{x^n}=\left(x+\frac{1}{x}\right)^n-n\left(x+\frac{1}{x}\right)^{n-2}+\frac{n(n-3)}{1.2}\left(x+\frac{1}{x}\right)^{n-4}-$$
$$\frac{n(n-4)(n-5)}{1.2.3}\left(x+\frac{1}{x}\right)^{n-6}+\text{ \&c.}$$

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